

Dissertation

Christoph Kißling

Circuit Design and Experimental Characterization of a Josephson Traveling-Wave Parametric Amplifier

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Dissertation

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Abstract

Bringing signals from quantum to classical, from few-photon to many-photon levels for further processing, is a common challenge in rising quantum technologies. These include quantum computing, quantum communication, quantum sensing, radio astronomy, and dark matter searches, typically operating at microwave frequencies and millikelvin temperatures. Superconducting parametric amplifiers, as the first piece in the readout chain, have become a cornerstone of these technologies, since they offer unrivaled noise performance near the quantum-mechanical limit and high gain to surpass the noise of subsequent amplifiers.

A notable implementation is the Josephson traveling-wave parametric amplifier (TWPA), which stands out due to its bandwidth of several Gigahertz and high saturation power, often larger than -100 dBm. These key properties come at the cost of complexity in design and fabrication: TWPA metamaterials often contain several thousands of Josephson junctions. Designing high-gain TWPAs involves three major challenges: ensuring impedance-matching, phase-matching of parametric amplification, and suppressing unwanted processes, all achievable through sub-wavelength dispersion engineering.

This thesis covers the circuit design and analysis, implementation, and experimental characterization of a dispersion-engineered TWPA based on radio-frequency superconducting interference devices (rf-SQUIDs). It focuses on a detailed investigation of dispersion-engineering techniques and their effects on amplifier performance, particularly through the use of time-domain circuit simulations to analyze the multimode dynamics of the circuit. The experimentally realized device features 18 dB of gain, an instantaneous bandwidth spanning from 2.2 to 6.7 GHz, a system noise of 2.5 quanta, and a saturation power of (-86.5 ± 4.0) dBm. With these characteristics, it is suitable for frequency-multiplexed readout of detector arrays or qubits. Its high saturation power, about 10 dB higher than other reported Josephson-junction-based TWPAs, allows increasing the number of qubits per readout line, or decreasing the intermodulation between frequency-multiplexed qubits, both crucial for scaling up quantum computation.

Kurzfassung

Das Verstärken von schwachen, wenige Photonen umfassenden Signalen ist eine Herausforderung verschiedener Quantentechnologien, die bei Mikrowellenfrequenzen und Millikelvin-Temperaturen arbeiten. Beispiele sind Quantencomputing, -kommunikation und -sensorik, Radioastronomie und die Suche nach Dunkler Materie. Supraleitende parametrische Verstärker, als erstes Glied der Ausleseketten, sind aufgrund ihres Rauschens nahe dem quantenmechanischen Limit zu einem Eckpfeiler dieser Technologien geworden.

Von besonderer Bedeutung ist der Josephson Traveling-Wave Parametric Amplifier (TWPA), ein Verstärkerkonzept, welches sich durch Bandbreiten mehrerer Gigahertz und hohe Sättigungsleistungen oft oberhalb von -100 dBm auszeichnet. Diese Schlüsseleigenschaften sind jedoch durch hohe Komplexität in Design und Fertigung erkauft, da TWPA-Metamaterialien oft aus mehreren tausend Josephson-Kontakten bestehen. Um hohe Verstärkung zu erreichen, sind drei zentrale Herausforderungen zu meistern: Impedanzanpassung, Phasenanpassung für parametrische Verstärkung und die Unterdrückung unerwünschter parametrischer Prozesse. Diese Anforderungen können durch Schaltungsmodifikation im Subwellenlängenbereich, bezeichnet als Dispersionsmodifikation, erreicht werden.

Die vorliegende Arbeit befasst sich mit Schaltungsentwurf und -analyse, sowie der experimentellen Implementierung und Charakterisierung eines dispersionsmodifizierten auf rf-SQUIDs (engl. radio-frequency superconducting quantum interference devices) basierenden TWPAs. Ein Schwerpunkt der Arbeit bildet die Untersuchung unterschiedlicher Techniken zur Dispersionsmodifikation und deren Auswirkungen auf Verstärkereigenschaften, insbesondere unter der Verwendung von Schaltungssimulationen im Zeitbereich, welche die multimodale Dynamik des Verstärkers berücksichtigen. Der experimentell realisierte Verstärker zeigt eine Verstärkung von 18 dB in einer 3 dB-Bandbreite zwischen 2,2 und 6,7 GHz, ein Gesamttrauschen von 2,5 Photonen und eine Sättigungsleistung von $(-86,5 \pm 4,0)$ dBm. Damit ist er hervorragend geeignet für das Auslesen von Detektorarrays oder Qubits. Die hohe Sättigungsleistung, die etwa 10 dB höher ist als die anderer Josephson-TWPAs, ermöglicht es, die Anzahl der Qubits pro Kanal zu erhöhen oder die Intermodulation zwischen frequenz-gemultiplexten Qubits zu verringern, beides entscheidend für die Hochskalierung von Quantencomputern.

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Introduction

“... a new field of research seems to emerge; amplifier circuits surely have the same importance for electrical investigations as the microscope has in optics. Since no clear limit for the attainable amplification has yet been identified, one could hope [...] to delve almost infinitely into the smallest details; the dream of ‘hearing the grass grow’ once again becomes tangibly real for humanity.

— Walter Schottky, 1918

By the time of this quote¹, quantum mechanics was still in its infancy, and neither the Johnson-Nyquist noise theorem had been discovered (although Schottky had already formulated similar ideas) [2, 3], nor the quantum-mechanical limit of noise in linear phase-preserving amplification [4, 5]; limits which to achieve in a practical measurement apparatus must have seemed nearly impossible. Today, about a century later, vast developments in microwave engineering and superconducting electronics have made this a reality. Microwave systems consisting of a cascade of amplifiers, where the first stage is a superconducting parametric amplifier, operated at millikelvin temperatures, have repeatedly demonstrated noise performance approaching the quantum-mechanical limit. The development of such a superconducting parametric amplifier for microwave frequencies is the subject of this thesis.

¹Translated from Ref. [1]: “Durch Hintereinanderschalten von Glühkathodenverstärkern ist es in den letzten Jahren gelungen, Wechselströme von äußerst geringer Amplitude wahrnehmbar und meßbar zu machen. Viele technische Probleme haben dadurch eine ruckweise Förderung erfahren, aber auch dem Forscher scheint sich ein neues Gebiet zu erschließen; die Verstärkerschaltungen haben für elektrische Untersuchungen sicher dieselbe Bedeutung wie in der Optik das Mikroskop. Da sich bisher noch keine deutliche Grenze für die erreichbare Verstärkung gezeigt hat, konnte man hoffen, durch genügenden Schutz störungsfreie Aufstellung usw. hier sozusagen bis zum unendlich Kleinen vorzudringen; der Traum vom „Gras wachsen hören“ stellte sich wieder einmal recht greifbar der Menschheit dar.”

1.1 Near-quantum-limited microwave measurement setups

” *Indeed, the last essential quantum-mechanical stage of a measuring apparatus is a high-gain amplifier; it produces an output that we can lay our grubby, classical hands on.*

— Carlton M. Caves, 1982

on the quantum limits on noise in amplifiers

Near-quantum-limited noise performance is required for fast measurements of signals with amplitudes corresponding to only a few photons. Prominent applications range from radio astronomy [6, 7] and the search for dark matter [8, 9, 10], to quantum sensing [11, 12], quantum communication [13, 14, 15], and quantum computing [16, 17, 18, 19, 20, 21]. For example, for quantum error correction algorithms [22, 23, 24], which are crucial for implementing practical quantum computers [25], it is important to read out quantum states of qubits much faster than their relaxation time, which is approaching the millisecond range for superconducting qubits [26, 27]. Neither is it feasible for practical applications to integrate over long times to compensate for an insufficient signal-to-noise ratio, nor is it viable to increase the readout power, as exceeding a certain threshold can cause measurement-induced state transitions of the qubit, hampering the error-correction protocol. This threshold power corresponds to an average population in the readout resonator of usually only a few photons [28, 29] for transmon qubits and up to a hundred photons [30, 31] for fluxonium qubits. For these reasons, decreasing the noise of readout systems as far as possible is of cornerstone importance, with the first amplifier of the readout chain playing a pivotal role.

Staying in the example of frequency-multiplexed qubit readout, a typical cryogenic measurement apparatus is schematized in Fig. 1.1. Noise levels of room-temperature electronics are huge, with a lower bound given by the Johnson-Nyquist noise with a power spectral density of $k_B T$, where k_B is the Boltzmann constant, and T the temperature. To bring this noise level down to that of quantum fluctuations, qubits are placed in a cryostat with a sufficiently low base temperature, $k_B T \ll \hbar \omega$, with $\hbar$ the reduced Planck constant, and ω the angular frequency of the qubit or the readout resonator. Very common are base temperatures $T \sim 20$ mK and frequencies in the range of $f = \omega/2\pi \sim 1\text{--}10$ GHz. The input line is heavily attenuated to bring the noise level of room temperature electronics down to the thermal noise levels at the various temperature stages of the cryostat, and eventually, down to the base-temperature noise level, which is ideally determined by vacuum fluctuations,

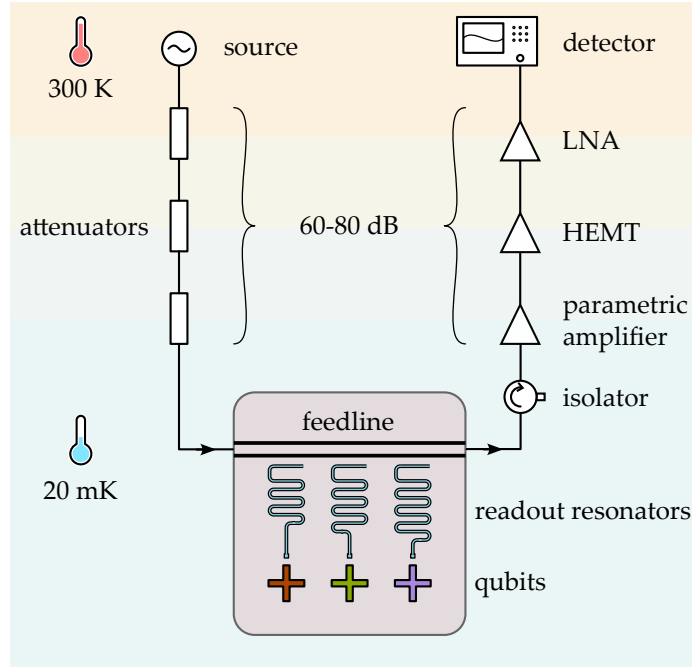


Fig. 1.1.: Typical measurement setup for superconducting qubits. Illustrated are transmon qubits with readout resonators coupled to a common feedline. Readout resonators have different resonance frequencies for frequency-multiplexed readout. Input attenuation decreases the input noise to the thermal noise level of the lowest temperature stage. Output amplification brings the few-photon powers of the readout tones to levels processable by room-temperature electronics. Essential for the system noise is the first amplifier in the chain, which is usually a superconducting parametric amplifier. It is followed by a cryogenic high-electron-mobility transistor (HEMT) amplifier and a room-temperature low-noise amplifier (LNA).

having a power spectral density corresponding to half a photon, $\hbar\omega/2$. In most practical superconducting qubit platforms, the qubit state is read out by measuring the state-dependent resonance shift of a readout resonator [32]. This protocol is called dispersive readout, and can be easily multiplexed in the frequency domain by varying the resonance frequencies of multiple readout resonators by design. The very weak readout tones then have to be amplified again to powers processable by room-temperature electronics – the ‘grubby, classical hands’.

For this purpose, a cascade of amplifiers is employed. The noise of a cascade of amplifiers is given by the Friis formula [33],

$$T_{N,\text{out}} = T_{N,1} + \frac{T_{N,2}}{G_1} + \frac{T_{N,3}}{G_1 G_2}, \quad (1.1)$$

where G_j is the power gain of the amplifier with index j , and $T_{N,j}$ is the noise temperature, an equivalent temperature that would produce the same amount of

Johnson-Nyquist noise as the amplifier adds to a signal. From this equation it follows that the first amplifier in the cascade has the highest impact on the total output noise, because the noise of subsequent amplifier stages is divided by the gain of the prior stages. Vast progress in the development of semiconductor-based microwave amplifiers has led to noise temperatures as low as 1.5 K in commercially available cryogenic High-Electron-Mobility-Transistor (HEMT) amplifiers [34]. This noise temperature corresponds to 8 quanta at 4 GHz and 4 quanta at 8 GHz. Lower added noise has only been demonstrated by superconducting parametric amplifiers, which offer noise performance approaching the quantum limit of half a quantum of added noise [35, 36, 37, 38].

1.2 Parametric amplifiers

Maybe the first device we would today call a parametric amplifier was based on the nonlinear inductance of a saturated iron core transformer and enabled radio-frequency transmission of telephone communication between Berlin and Vienna around 1914 [39, 40]. As a result of a pioneering work of van der Ziel in 1948 on the mixing properties of nonlinear capacitors [41], pointing out that these interactions can lead to ideally noise-free microwave amplification, and after the invention of low-loss silicon varactors – diodes with voltage-dependent capacitance –, a large number of experimental and theoretical works appeared in the 1950s, as reviewed in Ref. [40]. The gain of these devices, called reactance amplifiers or parametric amplifiers, is achieved by time-modulating (pumping) a nonlinear reactance, i.e., a nonlinear inductance or capacitance.

Parametric amplification can be illustrated on the example of a modulated nonlinear capacitance (Fig. 1.2a). Consider a capacitor made of two metal plates. When charged, the plates attract each other due to opposite charges, so it requires work to separate the plates. If they are separated to twice the original distance, the capacitance will halve, and the voltage will double, as the charge remains constant. Consequently, the electrostatic energy, which is proportional to the square of the voltage and directly proportional to the capacitance, will also double. The work used to separate the plates is now stored as additional electrostatic energy in the capacitor. Now, if this 'pumped' capacitor is combined with an inductor to form an oscillating circuit, the voltage on the capacitor will peak twice each cycle. If the capacitance is decreased (plates are separated) when the voltage is maximum and is increased when the voltage is zero, net energy will be added to the oscillation – the signal gets amplified.

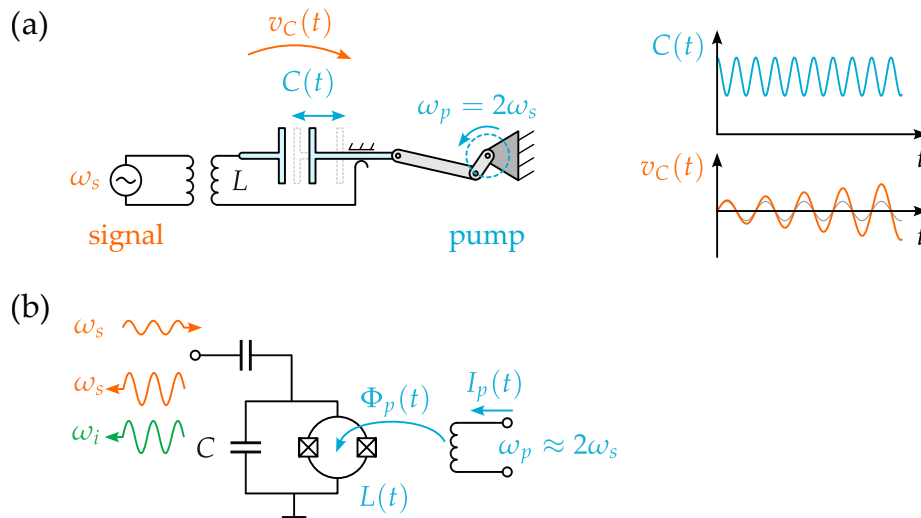


Fig. 1.2.: Illustration of the working principle of parametric amplification. **(a)** Resonant circuit with a periodically modulated (pumped) capacitance $C(t)$. When the 'pump' has the correct phase relative to the signal oscillation, the signal amplitude is amplified by periodically increasing the electrostatic energy. **(b)** The same is possible via modulating an inductance, e.g., via flux-pumping a SQUID. If the pump frequency is not twice the signal frequency, an idler oscillation $\omega_i = \omega_p - \omega_s$ is generated.

Note that parametric amplification in this example depends on the phase between the pump and the signal oscillations. If the signal oscillation is shifted by 90° , it gets deamplified rather than amplified. This regime is called phase-sensitive parametric amplification, occurring when the pump frequency is twice the signal frequency, $f_p = 2f_s$. However, parametric amplification is also possible for $f_p \neq 2f_s$. Since it is an energy-conserving process, an additional oscillation with frequency $f_i = f_p - f_s$ is generated, usually called the idler [42]. In this case, parametric amplification is not phase-sensitive, it is then called phase-preserving. Depending on the order of nonlinearity mediating parametric amplification, one can distinguish between three-wave mixing (3WM) and four-wave mixing (4WM), obeying the frequency relations $f_s + f_i = f_p$ and $f_s + f_i = 2f_p$, respectively. From a quantum-optics point of view, parametric amplification can be interpreted as the creation of a signal and an idler photon by the annihilation of one (two) pump photon(s) in 3WM (4WM). The additional signal photon adds to the signal amplitude, thereby amplifying.

Superconducting parametric amplifiers utilize modulated inductances instead of capacitances. An example, complementary to the pumped-capacitor circuit, is depicted in Fig. 1.2b. Here, the inductance of a Superconducting Quantum Interference Device (SQUID) is modulated via a time-periodic magnetic flux $\Phi_p(t)$,

acting as the pump. Again, it forms an oscillating circuit together with a (constant) capacitance. As an alternative to flux-pumping, which utilizes the flux-tunability of the SQUID inductance $L(\Phi)$, it is also possible to use the nonlinearity of the inductance $L(I)$, e.g., of a Josephson junction or a SQUID, and pump it with an injected pump current $I_p(t)$. In the same fashion, the inductance is then modulated and parametric amplification can be attained. First experiments on parametric amplification using Josephson junctions were performed in the late 1960s [43], but it took another two decades to obtain compelling performance [44, 35]. Due to the rise of quantum technologies in the past two decades, research on superconducting parametric amplifiers has seen increased interest. Superconducting parametric amplifiers have been realized in various material classes, based on the kinetic inductance of different disordered superconductors [38, 45, 46, 47] or nanowires [48], Josephson junctions [49], DC-SQUIDs [36, 50], radio frequency (rf) SQUIDs [44, 51], and Superconducting Nonlinear Asymmetric Inductive eLements (SNAILs) [52, 53].

1.3 Traveling-wave parametric amplifiers

A general distinction can be drawn between parametric amplifiers with resonant (standing-wave) architectures and with traveling-wave architectures [54, 55], both with the aim of achieving sufficiently long interaction times between the involved waves, and hence, sufficient power transfer for high gain. A traveling-wave parametric amplifier (TWPA), representing the subject of this thesis, is arranged as a transmission line of inductances and capacitances (Fig. 1.3), allowing for impedance-matching to exterior circuitry. Amplification occurs via the power transfer of a strong pump wave to a weak copropagating signal wave, and an idler wave is generated due to energy conservation.

Two major challenges govern the efforts in designing TWPAs with high gain. In 4WM, the main challenge is to ensure phase matching of the three (or four) waves involved in parametric amplification, all along the transmission line. Otherwise, the direction of power transfer reverses after a distance at which the accumulated phase mismatch equals π . The phase mismatch arises from nonlinear phase shifts, called self-phase and cross-phase modulation and is caused by a third-order nonlinearity [56]. In 3WM TWPAs based on rf-SQUIDs or SNAILs as nonlinear inductive elements, the third-order nonlinearity can be canceled via a constant magnetic field [57]. Parametric amplification is then facilitated by a second-order nonlinearity, devoid of self-phase and cross-phase modulation. However, due to their wideband nature, 3WM TWPAs facilitate a plethora of parametric processes, far beyond the

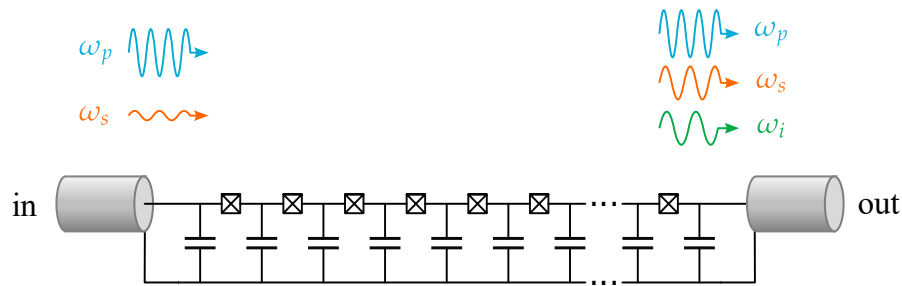


Fig. 1.3.: Traveling-wave parametric amplifier based on Josephson junctions. Capacitances to ground allow impedance-matching to external circuitry. The amplifier operates in transmission; amplification occurs via power transfer from the strong pump wave to a weak copropagating signal wave. The idler wave is generated due to energy conservation.

three primary waves [58]. These unintended processes cause power leakage to other tones, and thereby limit the attainable signal gain. For these reasons, special techniques, referred to as dispersion engineering [38, 59, 49, 50, 60], are essential to achieve high gain in a TWPA for both 3WM and 4WM.

1.4 Wishlist of amplifier figures of merit

Due to the role of superconducting parametric amplifiers as the first stage in an amplifier cascade, several figures of merit are of particular importance. First and maybe most important is the added noise of the amplifier. Nearly free of dissipation, superconducting parametric amplifiers have repeatedly demonstrated added noise approaching the quantum limit of half a quantum of noise [5]. Of the same importance is its gain; it should be high enough to overcome the noise of the subsequent amplifier stages. As an example, we may consider the noise temperature of the subsequent amplifier stages to be 3 K, and a quantum-limited parametric amplifier with a noise temperature of 100 mK at 4 GHz. Then, if the latter has a power gain of 100, or 20 dB, the input-referred system noise temperature is 0.13 K, only slightly above that of the parametric amplifier alone. A gain of 25–30 dB would further decrease the system noise, but may cause other issues, such as instability and increased gain ripples. Therefore, the targeted value for the parametric amplifier is 20 dB.

Further, the bandwidth and the saturation power are important figures of merit. To scale up quantum computers to a hundred qubits and beyond [61, 62], multiplexed use of hardware is a key enabler. This can be done, for example, via multiplexed dispersive qubit readout [63, 64, 65], coupling the readout resonators of several qubits to the same readout line (c.f. Fig. 1.1). The same applies for

multiplexed readout of detector arrays, for example in radio astronomy. Since multiplexing is commonly done in the frequency domain, a wide amplification bandwidth is required to allow some separation between the channels, minimizing crosstalk. Additionally, the power-handling capability of the amplification chain needs to be sufficiently high to read out the channels simultaneously. In contrast to resonator-based parametric amplifiers, in which a gain-bandwidth tradeoff is dictated by the resonance, TWPAs inherently feature wide bandwidths, typically in the range of several Gigahertz. Moreover, they come with potentially higher saturation powers than their resonant counterparts, which, however, have caught up in recent years [66, 67, 51]. Other figures of merit of a microwave amplifier are the flatness of the gain profile, and the backaction on the signal source. Last but not least, amplifiers can be further assessed by their power consumption, their resilience to various operating conditions, and, of course, their ease of use.

1.5 Thesis outline

This thesis describes the analysis, circuit design, implementation, and experimental characterization of an rf-SQUID-based dispersion-engineered 3WM TWPA, culminating in an amplifier with state-of-the-art characteristics. Special focus is placed on a comparison and detailed investigation of dispersion-engineering techniques and their effects on amplifier figures of merit, both with the help of circuit simulations and analytical modeling. The work presented here focuses on TWPAs primarily from the standpoint of electrical engineering. It therefore prefers a classical description rather than a quantum-mechanical one², the use of highly flexible simulation tools rather than rigorous analytical modeling, and emphasizes on circuit modeling including parasitic circuit elements, instrumental for further optimization.

This thesis is structured as follows. Chapter 2 gives an overview on superconducting nonlinear inductive elements, focussing on rf-SQUIDs, followed by a discussion of the linear and nonlinear properties of a homogeneous transmission line consisting of rf-SQUIDs, with the conclusion that dispersion engineering is essential to achieve high gain in a 3WM TWPA. Chapter 3 introduces the circuit simulation platform used for a comparison of dispersion-engineering techniques and three detailed case studies: on the effects of reflections in a TWPA, saturation power, and the impact of parameter spread on the performance of a TWPA. The findings of these analytical and simulation-based studies guide the way towards a practical implementation of a dispersion-engineered TWPA, described in Chapter 4. After presenting a detailed circuit model of the device, the protocol to extract

²For quantum-mechanical modeling of TWPAs refer to, e.g., Refs. [68, 69, 70, 71].

circuit parameters is described, followed by experiments on a parasitic process causing deamplification, and finally, presenting the characterization of the primary amplifier figures of merit, along with a comparison to the state of the art. As an outlook, the thesis discusses crucial tasks for future development of superconducting parametric amplifiers.

Three-wave mixing in rf-SQUID based homogeneous metamaterials

This chapter lays a theoretical foundation for rf-SQUID based three-wave-mixing TWPAs, originally devised by Zorin [57]. The description is restricted to the classical regime. A quantum-mechanical analysis is beyond the scope of this thesis and can be found, e.g., in Ref. [68].

The chapter begins with a detailed description of the rf-SQUID (radio-frequency superconducting quantum interference device), which is the central building block of the metamaterial, introducing nonlinearity and tunability to the circuit. Specifically, I extend the standard modeling of rf-SQUIDs to include nonidealities of our rf-SQUID implementation. Later on this proves crucial for understanding the behavior of real TWPA devices. Long chains of rf-SQUIDs, along with capacitors to ground, arranged in an LC ladder-line topology, form the backbone of the TWPA metamaterial design. Transmission-line theory provides linear properties of wave propagation in such a metamaterial. The subsequently presented coupled-mode theory describes the nonlinear coupling between copropagating waves. This theory not only allows insight into the underlying physics of parametric amplification but also offers closed-form expressions for central figures of merit of a parametric amplifier. Finally, an analysis of the limitations of coupled-mode theory motivates the use of a circuit simulator to rigorously analyze and further optimize TWPA circuits, setting the stage for the focus of the next two chapters in this thesis.

2.1 Superconducting nonlinear inductive elements

2.1.1 Josephson junction as a nonlinear inductive element

The cornerstone of circuit quantum electrodynamics and superconducting quantum computing is the Josephson junction. The Josephson effect [72] describes quantum-mechanical tunneling of charge carriers (Cooper pairs) through a barrier between two superconducting electrodes. A typical implementation is a sandwich structure with two superconducting electrodes separated by a thin (~ 1 nm) insulator (Fig. 2.1a) and is in the following called a Josephson junction. In this thesis, all Josephson junctions are fabricated in Niobium (Nb) thin-film technology as a Nb/Al-AlO_x/Nb sandwich structure, with oxidized Aluminum (AlO_x) as a tunnel barrier (Fig. 2.1b). Appendix A.2.1 presents an experimental characterization of these junctions, complementing their theoretical description provided here. Figure 2.1c shows the circuit symbol of a Josephson junction, described by the resistively-capacitively shunted junction model (RCSJ), which takes into account the quasiparticle resistance R_J and the Josephson capacitance C_J (Fig. 2.1d). In this section, only ideal Josephson junctions are considered, represented by the symbol in the center of Fig. 2.1d.

In the field of circuit quantum electrodynamics, a Josephson junction is typically operated in the zero-voltage state as an ideally lossless nonlinear inductive element. In this state, current $i(t)$ through and voltage $v(t)$ across a Josephson junction depend on φ , the gauge-invariant phase difference between the macro-

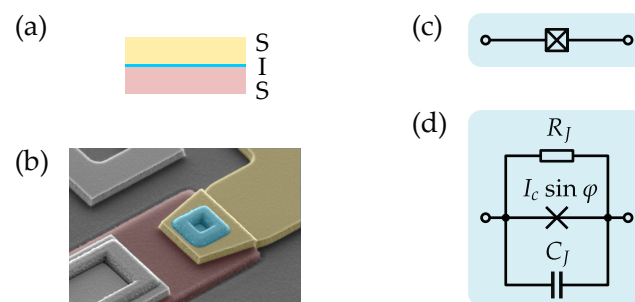


Fig. 2.1.: Josephson tunnel junction. **(a)** Scheme of a Josephson tunnel junction with two superconducting electrodes (S) separated by an insulator as tunnel barrier (I). **(b)** False-color micrograph of a physical implementation of a Josephson junction in Nb/Al-AlO_x/Nb trilayer technology, where yellow and red indicate the Nb layers and blue marks the area where the Nb layers are only separated by the AlO_x tunnel barrier, which is the active area of the Josephson junction. **(c)** Symbol of a Josephson junction. **(d)** Resistively-capacitively shunted junction model, where the symbol in the center represents an ideal Josephson junction. Scanning electron micrograph by Jan Blohm and Victor Gaydamachenko.

scopic quantum-mechanical wave functions of the two superconducting electrodes, according to the Josephson equations,

$$i(t) = I_c \sin \varphi(t), \quad (2.1)$$

$$\frac{\partial \varphi(t)}{\partial t} = \frac{2\pi}{\Phi_0} v(t). \quad (2.2)$$

Here, I_c denotes the critical current of the junction, which is the maximum supercurrent that can flow continuously before the junction switches to the voltage state, $\Phi_0 = h/2e$ is the magnetic flux quantum, h is the Planck constant, and e is the electron charge. For convenience, we introduce the reduced flux quantum $\varphi_0 = \Phi_0/(2\pi)$. When defining magnetic flux $\Phi = \varphi_0 \varphi$, the similarity of the second Josephson equation and Faraday's induction law becomes apparent,

$$v(t) = \varphi_0 \frac{\partial \varphi(t)}{\partial t} = \frac{\partial \Phi(t)}{\partial t}. \quad (2.3)$$

Consequently, the Josephson junction exhibits inductive behaviour,

$$\begin{aligned} v(t) &= \varphi_0 \frac{\partial \varphi}{\partial i} \frac{\partial i(t)}{\partial t} \\ &= L_J \frac{\partial i(t)}{\partial t} \end{aligned} \quad (2.4)$$

where L_J is the inductance of the Josephson junction. This inductive behavior arises from the phase-dependent tunneling of Cooper pairs, rather than from the energy of the magnetic field. The superconducting film itself also possesses a kinetic inductance, arising from the inertia of the Cooper pairs, which is further analyzed in Appendix A.2.2. For a phase difference $\varphi = \varphi_{dc} + \phi$ consisting of a constant phase φ_{dc} and a vanishingly small AC perturbation ϕ , we find the phase-biased small-signal inductance of the Josephson junction using Eq. 2.1,

$$\begin{aligned} L_J(\phi \rightarrow 0) &= \varphi_0 \left. \frac{\partial \varphi}{\partial i} \right|_{\varphi_{dc}} \\ &= \varphi_0 \left(\frac{\partial i(\varphi)}{\partial \varphi} \right)^{-1} \bigg|_{\varphi_{dc}} \\ &= \frac{\Phi_0}{2\pi I_c} \frac{1}{\cos \varphi_{dc}}. \end{aligned} \quad (2.5)$$

The small-signal inductance of an unbiased junction, $\phi_{dc} = 0$, is

$$L_{J0} = \frac{\Phi_0}{2\pi I_c}. \quad (2.6)$$

For small but not vanishing amplitudes of ϕ , the current-phase relation (Eq. 2.1) of an unbiased Josephson junction can be approximated as

$$\frac{i(\phi)}{I_c} \approx \phi - \frac{\phi^3}{6}. \quad (2.7)$$

Equivalently, the nonlinear Josephson inductance can be expressed as

$$\begin{aligned} L_J(i) &= \frac{\Phi_0}{2\pi I_c} \frac{1}{\cos \phi(i)} \\ &= \frac{\Phi_0}{2\pi I_c} \frac{1}{\sqrt{1 - (i/I_c)^2}} \\ &\approx \frac{\Phi_0}{2\pi I_c} \left(1 + \frac{1}{2} \left(\frac{i}{I_c} \right)^2 \right), \end{aligned} \quad (2.8)$$

The nonlinear inductance is analogous to the refractive index of a nonlinear optical medium, which depends on the light intensity $\propto \vec{E}^2$. Equivalently, the term $\propto \phi^3$ in Eq. 2.7 corresponds to the Kerr nonlinearity in nonlinear optics, where the polarization vector takes the form $\vec{P} \approx \epsilon_0(\chi^{(1)}\vec{E} + \chi^{(3)}\vec{E}^3)$. The current-phase relation is centrosymmetric, $i(-\phi) = -i(\phi)$, and its nonlinearity enables four-wave mixing (4WM), as utilized for example in Refs [36, 49]. Other effects facilitated by the Kerr nonlinearity, unwanted and detrimental in the context of parametric amplification, include self-phase and cross-phase modulation, and third-harmonic generation [56].

To achieve three-wave mixing (3WM), breaking of this symmetry is necessary, which is possible by biasing the junction at a constant phase ϕ_{dc} , such that the current-phase relation

$$i(\phi, \phi_{dc}) = I_c \sin(\phi_{dc} + \phi) \quad (2.9)$$

$$= I_c (\sin(\phi_{dc}) \cos(\phi) + \cos(\phi_{dc}) \sin(\phi)) \quad (2.10)$$

$$\approx I_c \left(\sin(\phi_{dc}) \left(1 - \frac{\phi^2}{2} + \dots \right) + \cos(\phi_{dc}) \left(\phi - \frac{\phi^3}{6} + \dots \right) \right) \quad (2.11)$$

contains noncentrosymmetric terms $\propto \phi^2$. This type of nonlinearity is analogous to a polarization vector including a term $\chi^{(2)}\vec{E}^2$ in nonlinear optics (e.g. in Lithium niobate, LiNbO₃), and enables 3WM. However, a consequence of Eqs. 2.5 and 2.11

is that a Josephson junction practically does not exhibit pure noncentrosymmetric nonlinearity, which is required for pure 3WM. This is because at $\phi_{dc} = \pi/2$, where the Kerr nonlinearity vanishes, the current approaches the critical current of the junction, and the linear inductance $L_J \propto 1/\cos(\phi_{dc})$ approaches infinity.

2.1.2 Nonhysteretic rf-SQUID as nonlinear inductive element

A strategy to achieve noncentrosymmetric nonlinearity, and hence pure 3WM, was introduced by Zorin [57], who proposed to use nonhysteretic rf-SQUIDs instead of bare Josephson junctions as nonlinear inductive elements. Parametric amplifiers based on rf-SQUIDs were proposed and realized already in the late 1980s by Yurke et al. [73, 44]. Alternatively, noncentrosymmetric nonlinearity can be achieved with asymmetric SQUIDs with several (more than one) large Josephson junctions in one branch of the SQUID, and one single small junction in the other branch. This configuration is called Superconducting Nonlinear Asymmetric Inductive eLement (SNAIL) [52, 74, 75] and exhibits nonlinear characteristics similar to those of an rf-SQUID. SNAILs have been used as a building block to construct TWPAs in Refs. [53, 76, 77]. In the following description, I broadly adhere to Zorin's theoretical framework as presented in Ref. [57].

One-junction SQUIDs

An rf-SQUID consists of a superconducting loop intersected by a Josephson junction, forming a one-junction superconducting interference device (SQUID). The arrangement of the junction in a closed loop causes interference between the superconducting wavefunctions of both sides of the junction (A and B in Fig. 2.2a). To obtain the interferometer equations, we integrate the gradient of the phase θ of

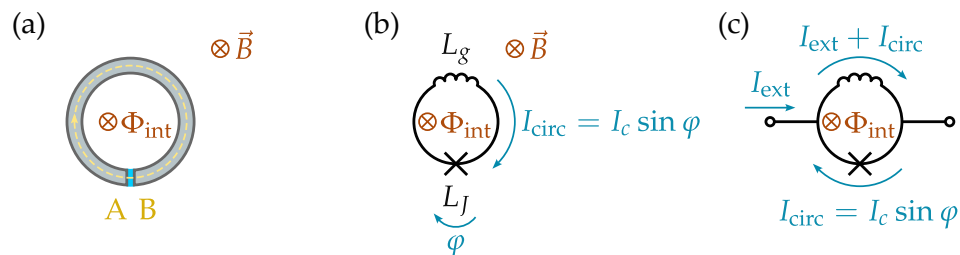


Fig. 2.2.: Sketch and equivalent circuit diagram of an rf-SQUID. **(a)** Superconducting loop intersected by a Josephson junction (blue) forms a one-junction SQUID. The magnetic field $\vec{B}$, pointing into the plane, causes a magnetic flux Φ_{int} in the loop. Dashed line: closed integration path. **(b)** Circuit diagram of an isolated rf-SQUID, having geometrical inductance L_g and Josephson inductance L_J . **(c)** rf-SQUID as two-terminal circuit element (one-port), with current I_{ext} flowing through.

the wavefunction along a closed path (dashed line in Fig. 2.2a). Continuity of the wavefunction demands

$$\oint \nabla \theta \cdot d\vec{l} = 2\pi n, \quad (2.12)$$

where n is an integer. In a superconductor, θ is linked to the magnetic vector potential $\vec{A}$ and the superconducting current density $\vec{J}_S$ via the relation $(\Phi_0/2\pi)\nabla\theta = \mu_0\lambda_L^2\vec{J}_S + \vec{A}$ [78], where λ_L denotes the penetration depth. Considering an integration path deep inside a thick superconductor, $\vec{J}_S$ is negligible and we can substitute $\nabla\theta = (2\pi/\Phi_0)\vec{A}$. Splitting the integration path in two parts, the segment from A to B along the superconductor results in

$$\int_A^B \nabla \theta \cdot d\vec{l} = \frac{2\pi}{\Phi_0} \int_A^B \vec{A} \cdot d\vec{l}. \quad (2.13)$$

To get the contribution of the integration path from B to A across the Josephson junction, we introduce the gauge-invariant phase difference [78]

$$\varphi = \int_B^A \left(\nabla \theta - \frac{2\pi}{\Phi_0} \vec{A} \right) \cdot d\vec{l}. \quad (2.14)$$

Rearranging Eq. 2.14 and then closing the integration path again using Eq. 2.13 yields

$$\oint \nabla \theta \cdot d\vec{l} = \varphi + \frac{2\pi}{\Phi_0} \oint \vec{A} \cdot d\vec{l}. \quad (2.15)$$

By Stokes' theorem, the closed integral over $\vec{A}$ equals the magnetic flux Φ_{int} threading the loop. Then, substituting Eq. 2.12 into Eq. 2.15 relates φ to Φ_{int} ,

$$\begin{aligned} \varphi - 2\pi n &= -\frac{2\pi}{\Phi_0} \oint \vec{A} \cdot d\vec{l} \\ &= -\frac{2\pi\Phi_{\text{int}}}{\Phi_0}, \end{aligned} \quad (2.16)$$

This phase difference across the Josephson junction determines a current circulating in the loop

$$\begin{aligned} I_{\text{circ}} &= I_c \sin \varphi \\ &= I_c \sin \left(-\frac{2\pi\Phi_{\text{int}}}{\Phi_0} + 2\pi n \right) \\ &= -I_c \sin \left(\frac{2\pi\Phi_{\text{int}}}{\Phi_0} \right). \end{aligned} \quad (2.17)$$

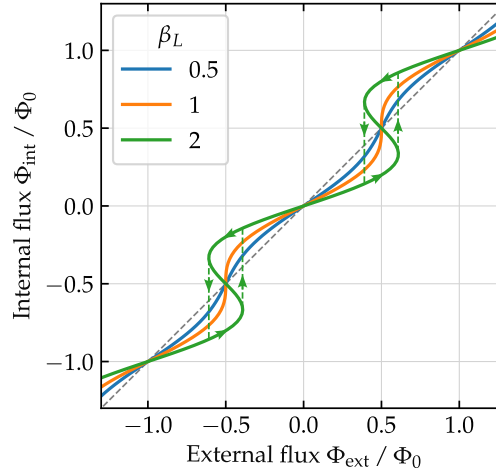


Fig. 2.3.: Dependence of the total magnetic flux on externally applied flux in an rf-SQUID. The rf-SQUID exhibits hysteresis when the screening parameter $\beta_L > 1$.

In the presence of an external magnetic field $\vec{B}$, the magnetic flux in the loop has two contributions: the external flux Φ_{ext} , and the flux $L_g I_{\text{circ}}$ generated by the circulating current, where L_g denotes the geometrical inductance of the SQUID loop. Hence, we find the transcendental equation

$$\begin{aligned}\Phi_{\text{int}} &= \Phi_{\text{ext}} + L_g I_{\text{circ}} \\ &= \Phi_{\text{ext}} - L_g I_c \sin\left(\frac{2\pi\Phi_{\text{int}}}{\Phi_0}\right).\end{aligned}\quad (2.18)$$

Introducing the rf-SQUID screening parameter

$$\beta_L = \frac{2\pi L_g I_c}{\Phi_0} \quad (2.19)$$

as the ratio of geometrical inductance L_g and Josephson inductance L_{J0} (Eq. 2.6), we can reformulate Eq. 2.18 as

$$\Phi_{\text{int}} + \frac{\Phi_0 \beta_L}{2\pi} \sin\left(\frac{2\pi\Phi_{\text{int}}}{\Phi_0}\right) = \Phi_{\text{ext}}, \quad (2.20)$$

where Φ_{ext} is the independent variable. Figure 2.3 illustrates this dependence for different values of β_L . As Φ_{ext} increases, the circulating current creates a flux opposing Φ_{ext} , causing Φ_{int} to increase slower than Φ_{ext} . After exceeding $\Phi_{\text{int}} = \Phi_0/4$, where I_{circ} reaches its maximum and approaches I_c , the circulating current decreases again and reverses direction at $\Phi_{\text{int}} = \Phi_0/2$. In this way, I_{circ} never exceeds the critical current I_c . For $\beta_L < 1$, the rf-SQUID is nonhysteretic, uniquely

mapping Φ_{ext} onto Φ_{int} , while for $\beta_L > 1$, it behaves hysteretically around half flux quanta. The remainder of this work focuses on nonhysteretic rf-SQUIDs.

Nonhysteretic rf-SQUID as circuit element

Up to this point we have treated the rf-SQUID as an isolated arrangement (Fig. 2.2b). The following section describes the rf-SQUID used as a one-port circuit element, as schematized in Fig. 2.2c. When a current I_{ext} flows into the node of the rf-SQUID, current conservation demands that the current through the inductor L_g is $I_{\text{ext}} + I_{\text{circ}}$. This results in an additional term $L_g I_{\text{ext}}$ in Eq. 2.20,

$$\Phi_{\text{int}} + \frac{\Phi_0 \beta_L}{2\pi} \sin\left(\frac{2\pi \Phi_{\text{int}}}{\Phi_0}\right) = \Phi_{\text{ext}} + L_g I_{\text{ext}}. \quad (2.21)$$

It can be seen that Φ_{ext} and $L_g I_{\text{ext}}$ are equivalent in their effect on Φ_{int} . A remarkable consequence is that for any applied current I_{ext} , $|I_{\text{circ}}| \leq I_c$. Hence, the Josephson junction in the rf-SQUID is protected from switching to the voltage state. This is in contrast to other nonlinear inductive elements such as a single Josephson junction (Sec. 2.1.1) or a SNAIL, which both have a maximum supercurrent, determined by the critical currents of the involved Josephson junctions. Upon exceeding this maximum supercurrent, these devices switch to the voltage state.

The voltage drop on the rf-SQUID, defined in the same direction as the current I_{ext} , using Eqs. 2.2 and 2.16, is

$$v_{\text{SQ}}(t) = -\frac{\Phi_0}{2\pi} \frac{\partial \varphi}{\partial t} = \frac{\partial \Phi_{\text{int}}}{\partial t}. \quad (2.22)$$

Due to the vanishing DC resistance of the superconducting shunt implementing the inductance L_g , only time-varying currents or fluxes result in a voltage drop across the rf-SQUID.

Small-signal inductance of a flux-biased rf-SQUID

For a constant external flux $\Phi_{\text{ext,dc}}$, generated either by an external magnetic field or by a constant current I_{dc} , the constant internal flux Φ_{dc} is the root of the transcendental equation

$$\Phi_{\text{dc}} + \frac{\Phi_0 \beta_L}{2\pi} \sin\left(\frac{2\pi \Phi_{\text{dc}}}{\Phi_0}\right) = \Phi_{\text{ext,dc}}, \quad (2.23)$$

which can be solved numerically. For convenience, we rewrite this flux as a phase drop $\phi_{\text{dc}} = 2\pi \Phi_{\text{dc}} / \Phi_0 = \Phi_{\text{dc}} / \varphi_0$. Therefore, the phase drop across the junction is $\varphi = -\phi_{\text{dc}}$, and the biased small-signal inductance of the junction, $L_J(\phi_{\text{dc}})$, is given

by Eq. 2.5. Calculating $L_{\text{SQ}}^{-1} = L_g^{-1} + L_J^{-1}$ yields the small-signal inductance of the rf-SQUID,

$$L_{\text{SQ}}(\phi_{\text{dc}}) = \frac{L_g}{1 + \beta_L \cos \phi_{\text{dc}}}. \quad (2.24)$$

The dependence on the flux bias is illustrated in Fig. 2.4a for two different values of β_L and $L_g = 100$ pH. Flux-tunability of the rf-SQUID increases with increasing β_L .

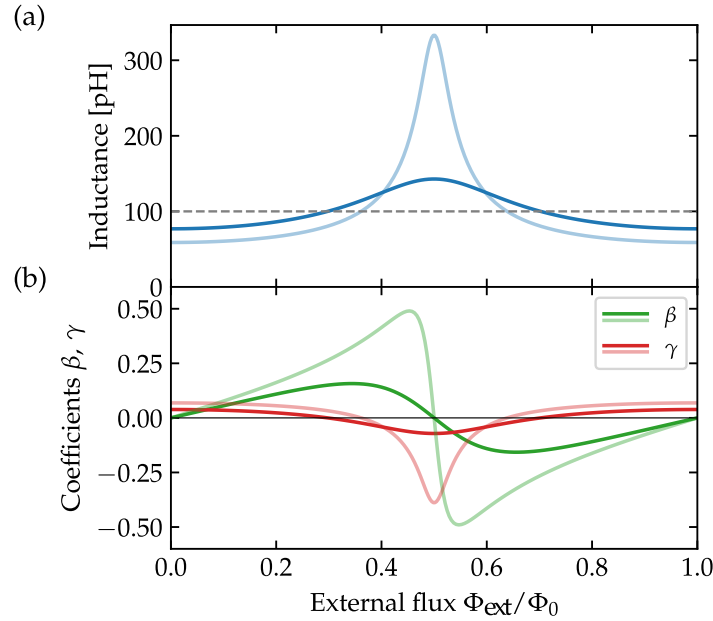


Fig. 2.4.: Inductance and nonlinear coefficients of a flux-biased rf-SQUID with $\beta_L = 0.3$ (dark lines) and 0.7 (light-colored lines). **(a)** Small-signal inductance. The dashed line marks the value of $L_g = 100$ pH. **(b)** Nonlinearity coefficients of second order (β) and third order (γ) of the current-phase relation.

Nonlinear response of a flux-biased rf-SQUID

To account for the nonlinear response of the rf-SQUID, we consider an injected current $i(t)$ superposed with the flux bias, $\Phi_{\text{ext}} = \Phi_{\text{ext,dc}} + L_g i(t)$. The flux $\Phi_{\text{int}} = \varphi_0(\phi_{\text{dc}} + \phi(t))$ is the nonlinear response of the rf-SQUID, where the phase $\phi(t)$ is associated with $i(t)$. Inserting Φ_{int} and Φ_{ext} into Eq. 2.20,

$$\varphi_0 \phi_{\text{dc}} + \varphi_0 \phi(t) + \varphi_0 \beta_L \sin(\phi_{\text{dc}} + \phi(t)) = \Phi_{\text{ext,dc}} + L_g i(t), \quad (2.25)$$

and substituting $\Phi_{\text{ext,dc}}$ using Eq. 2.23, we obtain the current-phase relation of the flux-biased rf-SQUID

$$\begin{aligned} i(t) &= \frac{I_c}{\beta_L} \phi(t) + I_c [\sin(\phi_{\text{dc}} + \phi(t)) - \sin \phi_{\text{dc}}] \\ &= \frac{I_c}{\beta_L} \phi(t) + I_c [\sin \phi_{\text{dc}} \cos \phi(t) + \cos \phi_{\text{dc}} \sin \phi(t) - \sin \phi_{\text{dc}}], \end{aligned} \quad (2.26)$$

where a trigonometric identity was used.

In the case of an inserted current with a small amplitude $|i| \ll \Phi_0/L_g$, it is useful to expand Eq. 2.26 into a series of the small phase $\phi = \phi(t)$,

$$\begin{aligned} i(t) &= \frac{I_c}{\beta_L} \phi + I_c \left[\sin(\phi_{\text{dc}}) \left(1 - \frac{\phi^2}{2} + \dots \right) + \cos(\phi_{\text{dc}}) \left(\phi - \frac{\phi^3}{6} + \dots \right) - \sin(\phi_{\text{dc}}) \right] \\ &= I_c \left[\left(\frac{1}{\beta_L} + \cos \phi_{\text{dc}} \right) \phi - \frac{\sin \phi_{\text{dc}}}{2} \phi^2 - \frac{\cos \phi_{\text{dc}}}{6} \phi^3 + \dots \right]. \end{aligned} \quad (2.27)$$

We identify the linear term $\propto \phi$ as mediated by the small-signal inductance L_{SQ} , Eq. 2.24, by which we normalize the nonlinear terms, and obtain

$$i(\phi) = \frac{\varphi_0}{L_{\text{SQ}}(\phi_{\text{dc}})} [\phi - \beta \phi^2 - \gamma \phi^3 + \dots] \quad (2.28)$$

with the coefficients of second and third order nonlinearity,

$$\beta = \frac{\beta_L}{2} \frac{\sin \phi_{\text{dc}}}{1 + \beta_L \cos \phi_{\text{dc}}}, \quad (2.29)$$

$$\gamma = \frac{\beta_L}{6} \frac{\cos \phi_{\text{dc}}}{1 + \beta_L \cos \phi_{\text{dc}}}. \quad (2.30)$$

The flux-tunability of these coefficients, together with that of the small-signal inductance, is shown in Fig. 2.4. The third-order (Kerr) nonlinearity with coefficient γ is zero at $\phi_{\text{dc}} = \pi/2$, whereas the quadratic nonlinearity coefficient β is maximal at $\phi_{\text{dc}} \approx \pi/2 + \beta_L$.

Figure 2.5 illustrates the validity range of the series expansion to third order in ϕ , Eq. 2.28, for three values of ϕ_{dc} . For $\phi_{\text{dc}} = 0$ and $\phi_{\text{dc}} = \pi$, the Kerr coefficient γ is positive and negative, respectively, and the second-order nonlinearity vanishes. Therefore, the current-phase relation is centrosymmetric in these points (red dashed lines) and, for $\phi_{\text{dc}} = 0$, of the same form as that of a single Josephson junction, Eq. 2.10.

The main difference of the current-phase relation of the rf-SQUID compared to that of a Josephson junction is the additional term $I_c \phi / \beta_L = \varphi_0 \phi / L_g$ in Eq. 2.27. This term, which is independent of ϕ_{dc} , enables operation of the rf-SQUID at

$\phi_{dc} = \pi/2$, where the Kerr nonlinearity γ vanishes without diverging of the linear inductance (center panel in Fig. 2.5). The additional term arises from the inductance L_g shunting the diverging inductance of the Josephson junction (Eq. 2.5), yielding $L_{SQ}(\phi_{dc} = \pi/2) = L_g$. Effectively, at this flux point, the geometrical inductance entirely determines the linear properties, whereas the Josephson junction of the rf-SQUID contributes only its nonlinearity. In this way, an rf-SQUID can achieve a current-phase relation that exhibits Kerr-free, noncentrosymmetric nonlinearity, thus enabling pure three-wave mixing.

Large-signal response

Figure 2.5 also illustrates the consequences of the nonlinear response to an injected current $i(t)$ in these three bias points. The nonlinear response $\phi(t)$ (orange curves) deviates notably from the linear response $\phi_{lin} = L_{SQ}i(t)/\varphi_0$ (dashed green curves). Specifically, $\phi > \phi_{lin}$ for $\gamma > 0$ ($\phi_{dc} = 0$) and $\phi < \phi_{lin}$ for $\gamma < 0$ ($\phi_{dc} = \pi$), which is equivalent to an effective large-signal inductance $\bar{L}_{SQ} > L_{SQ}$ and $\bar{L}_{SQ} < L_{SQ}$, respectively. While the former is equivalent to the amplitude dependence of the refractive index of nonlinear optical crystals, causing e.g. self-focusing of optical beams [56], the latter does not have a known optical analogue [53]. This amplitude dependence of the inductance is an illustrative explanation of the aforementioned Kerr effect of self-phase modulation, imposing one of the main challenges in the development of 4WM TWPAs.

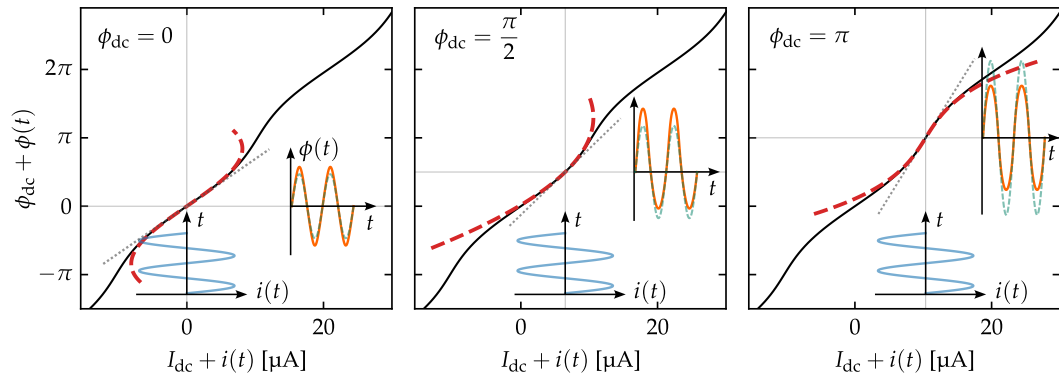


Fig. 2.5.: Nonlinear response to an injected current at different flux bias values. The flux bias values correspond to a positive (left), zero (center), and negative (right) Kerr coefficient. The current-phase relation (black line, Eq. 2.25) is approximated linearly (dotted line) or to third order (dashed red line, Eq. 2.28) around ϕ_{dc} . The slope of the dotted line is $L_{SQ}(\phi_{dc})$. An injected current $i(t)$ (blue curve) causes a nonlinear response $\phi(t)$ (orange curve), deviating from the linear response $L_{SQ}i(t)/\varphi_0$ (dashed green curve). This deviation has substantial consequences for phase matching in 4WM TWPAs. Parameters of the rf-SQUID are $L_g = 100$ pH and $\beta_L = 0.4$. The current amplitude is $7 \mu\text{A}$ and causes a phase response with an amplitude of ca. 2.1 rad in the Kerr-free case.

In the Kerr-free regime ($\phi_{\text{dc}} = \pi/2$), $\phi \approx \phi_{\text{lin}}$ when ignoring the non-zero mean value of $\phi(t)$, and thus $\bar{L}_{\text{SQ}} \approx L_{\text{SQ}}$. The non-zero mean arises from rectification of $i(t) \propto \sin(\omega t)$ by even-order terms $i^2, i^4, \dots$ in an expansion $\phi(i)$, yielding terms $\sin^2(\omega t), \sin^4(\omega t), \dots$ having a non-zero time-independent component. The associated amplitude-dependent constant phase ϕ_{rect} slightly shifts the bias point $\phi_{\text{dc}}^{\text{eff}} = \phi_{\text{dc}} + \phi_{\text{rect}}$, which has a secondary effect on the inductance $\bar{L}_{\text{SQ}}(\phi_{\text{dc}}^{\text{eff}})$. Together with deviations arising from higher-order nonlinearities $\propto \phi^4, \phi^5, \dots$, this rectification shift causes a small amplitude dependence of the effective large-signal inductance, which is experimentally investigated in Sec. 4.3. Given that amplitude dependence in linear properties, such as inductance, phase velocity, and impedance, is typically associated with the Kerr nonlinearity, we can conclude that a regime entirely free of ‘effective’ Kerr effects does not exist in current- or flux-biased rf-SQUIDs, despite a third-order nonlinearity coefficient equal to zero.

2.1.3 Nonideal rf-SQUIDs

In the preceding section, we considered an ideal rf-SQUID as circuit element (c.f. Fig. 2.2c), where all geometrical inductance L_g resides in one branch of the SQUID, with only the Josephson junction situated in the other arm. Moreover, any inductance in the leads attached to the rf-SQUID was neglected. However, in a realistic rf-SQUID, such as the implementation discussed in this thesis (Fig. 2.6), both the inductance of the leads and the inductance in the Josephson junction arm can have significant influence on the linear and nonlinear properties of the SQUID. Another difference between the ideal rf-SQUID and our implementations is the presence of a second, much larger, Josephson junction in the junction branch of the SQUID, which is used to swap metal layers (c.f. Fig. 2.2a and Fig. 4.1e,f). Such an rf-SQUID implementation is called a quasi-one-junction SQUID (QOJS) [79], in contrast to the ideal one-junction SQUID.

In this section, I extend the model of the ideal rf-SQUID (Sec. 2.1.2) to account for these nonidealities. The resulting model is applied to fit experimental data, as described in Sections 4.2.4, 4.3, and 4.4, in which the data could only be satisfactorily explained after including these nonidealities.

Nonideal rf-SQUID with parasitic inductance

First, we consider the presence of parasitic inductance in the Josephson junction branch of the rf-SQUID. The inductances in both branches, L_m and L_p , result in a total loop inductance $L_{\text{loop}} = L_m + L_p$. Hence, Eqs. 2.16–2.20 remain valid for this arrangement via substituting L_g by L_{loop} . The screening parameter is therefore defined as $\beta_L = 2\pi L_{\text{loop}} I_c / \Phi_0$. The parameter $\alpha_p = L_p / L_{\text{loop}}$ defines which fraction

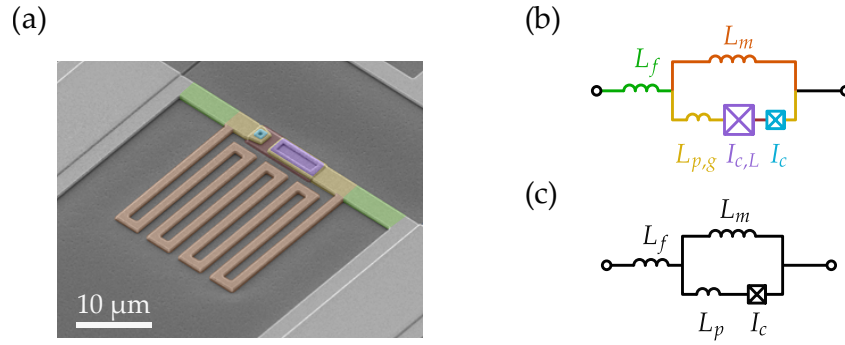


Fig. 2.6.: Quasi-one-junction SQUID with parasitic inductances. **(a)** False-color micrograph of a physical implementation of an rf-SQUID and, **(b)** its circuit representation. The large Josephson junction serves as a via, connecting the two metal layers (yellow and red). **(c)** Simplified schematic, where the small-signal inductance of the large Josephson junction is combined with the parasitic inductance $L_{p,g}$. In the presented rf-SQUID implementation, $L_p \approx L_f \approx L_m/2$ (refer to Tab. 4.3). Scanning electron micrograph by Jan Blohm and Victor Gaydamachenko.

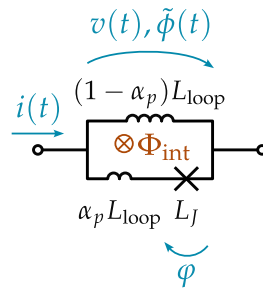


Fig. 2.7.: Nonideal rf-SQUID with parasitic inductance in the Josephson junction branch.

of L_{loop} is located in the Josephson junction branch of the SQUID (Fig. 2.7), and is a measure of nonideality. The small-signal inductance can be calculated as $L_{\text{SQ}}(\phi_{\text{dc}})^{-1} = L_m^{-1} + (L_p + L_J(\phi_{\text{dc}}))^{-1}$, resulting in

$$L_{\text{SQ}}(\phi_{\text{dc}}) = L_m \frac{1 + \alpha_p \beta_L \cos \phi_{\text{dc}}}{1 + \beta_L \cos \phi_{\text{dc}}}, \quad (2.31)$$

using the Josephson inductance, Eq. 2.5. The constant phase ϕ_{dc} is determined by Eq. 2.23 and can be set by an external magnetic field, or by injecting a current I_{dc} , such that $\Phi_{\text{ext,dc}} = (1 - \alpha_p)L_{\text{loop}}I_{\text{dc}}$. When a current $i(t)$ is also injected, the associated nonlinear response $\phi(t)$ is given by the relation

$$I_{\text{dc}} + i(t) = \frac{\varphi_0}{(1 - \alpha_p)L_{\text{loop}}} [\phi_{\text{dc}} + \phi(t) + \beta_L \sin(\phi_{\text{dc}} + \phi(t))], \quad (2.32)$$

which is similar to Eq. 2.25. However, an important difference to the ideal rf-SQUID is that the circulating current in the SQUID creates a phase drop on the parasitic

inductance $\alpha_p L_{\text{loop}}$. Consequently, the voltage drop across the SQUID, $v(t) = \varphi_0 \partial \tilde{\phi}(t) / \partial t$, is determined by the phase drop $\tilde{\phi}(t)$, which is not equal to $\phi(t) = -\varphi(t)$ (Fig. 2.7). To expand Eq. 2.32 into a Taylor series in $\tilde{\phi}$, it is therefore necessary to take into account the dependence $\tilde{\phi}(\phi)$. This is carried out in Appendix A.1.1 and yields

$$i(\tilde{\phi}) = \frac{\varphi_0}{L_{\text{SQ}}(\phi_{\text{dc}})} [\tilde{\phi} - \beta \tilde{\phi}^2 - \gamma \tilde{\phi}^3 + \dots], \quad (2.33)$$

with the modified coefficients of second-order and third-order nonlinearity,

$$\beta = \frac{\beta_L(1 - \alpha_p)}{2} \frac{\sin \phi_{\text{dc}}}{(1 + \beta_L \cos \phi_{\text{dc}})(1 + \alpha_p \beta_L \cos \phi_{\text{dc}})^2}, \quad (2.34)$$

$$\gamma = \frac{\beta_L(1 - \alpha_p)}{6} \frac{(1 + \alpha_p \beta_L \cos \phi_{\text{dc}}) \cos \phi_{\text{dc}} + 3\alpha_p \beta_L \sin^2 \phi_{\text{dc}}}{(1 + \beta_L \cos \phi_{\text{dc}})(1 + \alpha_p \beta_L \cos \phi_{\text{dc}})^4}. \quad (2.35)$$

In the ideal case $\alpha_p = 0$, the equations are identical to Eqs. 2.29 and 2.30, as expected. The flux dependences of the small-signal inductance and the nonlinearity coefficients are shown in Fig. 2.9.

Quasi-one-junction SQUID

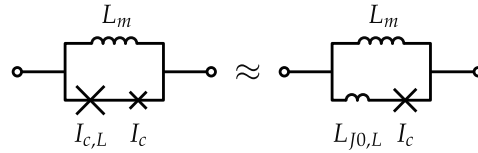


Fig. 2.8.: A quasi-one-junction SQUID and its approximation for $I_{c,L} \gg I_c$.

A quasi-one-junction SQUID is a two-junction SQUID, where one junction is much larger than the other, $I_{c,L} \gg I_c$, and both junctions are in the same branch of the SQUID loop (Fig. 2.8). The other branch of the SQUID consists of a geometrical inductance L_m , where 'geometrical' is used to distinguish it from Josephson inductances, but L_m also contains a kinetic inductance contribution (see Sec. 4.2.1). It is shown in Appendix A.1.2 that a quasi-one-junction SQUID is equivalent to a non-ideal rf-SQUID (one-junction SQUID) with a parasitic linear inductance $L_p = L_{J0,L}$ in the junction branch (Fig. 2.8), and is therefore described by Eqs. 2.31–2.35.

If the quasi-one-junction SQUID also contains parasitic geometrical inductance $L_{p,g}$, as schematized above in Fig. 2.6b, it can be combined with the inductance of the large junction to a total parasitic inductance $L_p = L_{p,g} + L_{J0,L}$ (Fig. 2.6c). Then, the loop inductance is $L_{\text{loop}} = L_m + L_{p,g} + L_{J0,L}$. With the parameters

$$\alpha_p = \frac{L_{p,g} + L_{J0,L}}{L_m + L_{p,g} + L_{J0,L}}, \quad (2.36)$$

and

$$\beta_L = \frac{2\pi I_c (L_m + L_{p,g} + L_{J0,L})}{\Phi_0}, \quad (2.37)$$

the Eqs. 2.31–2.35 describe the linear and nonlinear behaviour of the SQUID.

Nonideal rf-SQUID with lead inductance

Finally, we include the inductance L_f of the leads attached to the rf-SQUID to the model. Then, the small-signal inductance is

$$L_{SQ}(\phi_{dc}) = L_f + L_m \frac{1 + \alpha_p \beta_L \cos \phi_{dc}}{1 + \beta_L \cos \phi_{dc}}, \quad (2.38)$$

with α_p and β_L as defined in Eqs. 2.36 and 2.37, respectively. The phase drop $\hat{\phi}(t)$ across the nodes relates to an injected current $i(t)$ via

$$i(\hat{\phi}) = \frac{\varphi_0}{L_{SQ}(\phi_{dc})} [\hat{\phi} - \beta \hat{\phi}^2 - \gamma \hat{\phi}^3 + \dots], \quad (2.39)$$

with the second-order and third-order nonlinearity coefficients,

$$\beta = \frac{p \beta_L (1 - \alpha_f)}{2} \frac{\sin \phi_{dc}}{(1 + \beta_L \cos \phi_{dc})(1 + \alpha_f \beta_L \cos \phi_{dc})^2}, \quad (2.40)$$

$$\gamma = \frac{p^2 \beta_L (1 - \alpha_f)}{6} \frac{(1 + \alpha_f \beta_L \cos \phi_{dc}) \cos \phi_{dc} + 3 \alpha_f \beta_L \sin^2 \phi_{dc}}{(1 + \beta_L \cos \phi_{dc})(1 + \alpha_f \beta_L \cos \phi_{dc})^4}, \quad (2.41)$$

see Appendix A.1.3. Here, $\alpha_f = (\alpha_p L_m + L_f) / (L_m + L_f)$, and $p = L_m / (L_m + L_f)$ denotes the inductance participation ratio at $\phi_{dc} = \pi/2$, indicating the proportion of the SQUID inductance to the total inductance.

The effect of the parasitic inductances L_f and L_p is illustrated in Fig. 2.9. For constant inductance L_m , the presence of L_p increases the loop inductance, which changes the shape of $L_{SQ}(\Phi_{ext})$. Furthermore, it also enhances the nonlinearity of the SQUID by increasing β_L , and the additional term $\propto \sin^2 \phi_{dc}$ in Eq. 2.35, compared to Eq. 2.30, leads to a notable shift of the Kerr-free point $\gamma = 0$ away from $\phi_{dc} = \pi/2$. The feedline inductance L_f reduces the participation ratio p , which has the effect of diluting the nonlinearity of the rf-SQUID, apart from increasing the linear inductance.

This model treats the rf-SQUID implementation, including nonideal elements, depicted with the equivalent circuit in Fig. 2.6, as a single lumped nonlinear inductive element. In Sections 4.2.4, 4.3, and 4.4, we utilize the distinct shape of the flux modulation $L_{SQ}(I_{dc})$, described by Eq. 2.38, to extract the whole parameter

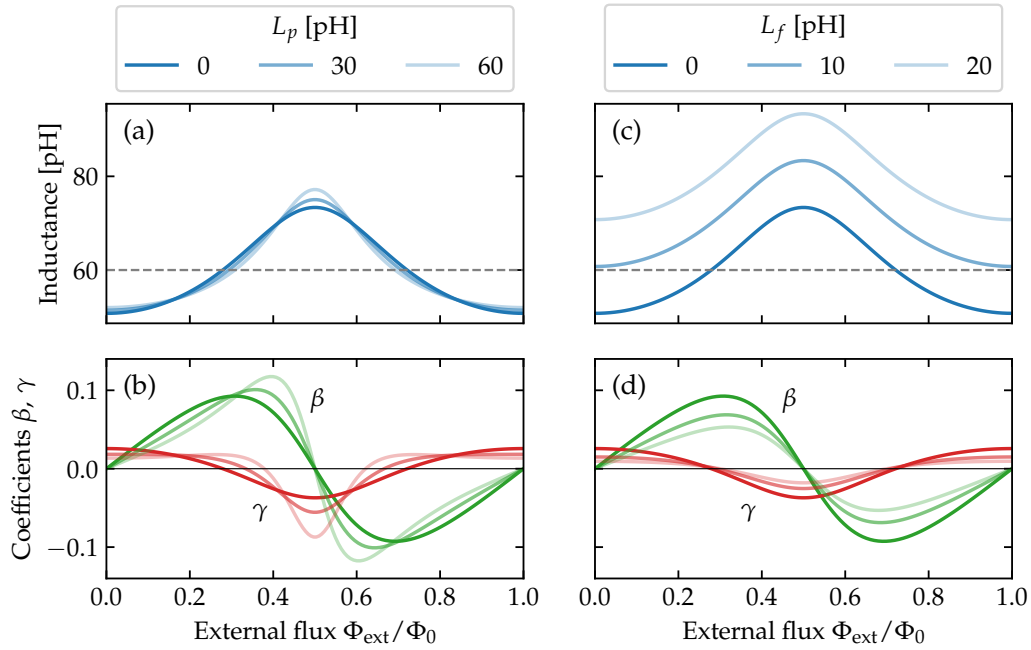


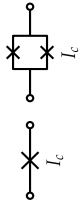
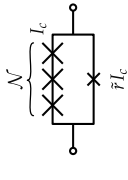
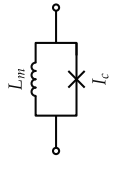
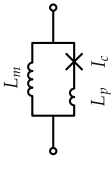
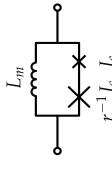
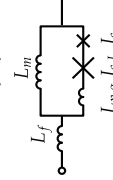
Fig. 2.9.: Small-signal inductance and nonlinear coefficients of a flux-biased nonideal rf-SQUID with parasitic inductances. **(a), (b)** Influence of the parasitic inductance L_p in the Josephson junction branch of the SQUID, while $L_f = 0$. **(c), (d)** Effect of the feedline inductance L_f , with $L_p = 0$. The presence of L_p increases the nonlinearity, while L_f increases the total inductance but decreases the nonlinearity of the SQUID. Parameters are $L_m = 60$ pH and $I_c = 1$ μ A in both cases.

set $\{L_m, L_p, L_f, I_c\}$ of the rf-SQUID. This plays a crucial role in the modeling and further analysis of our TWPA devices, as detailed in Sec. 4.

2.1.4 Overview on Josephson-junction-based nonlinear inductive elements

Summarizing this section, an overview of Josephson-junction-based nonlinear inductive elements is provided in Tab. 2.1. Their linear and nonlinear behavior is characterized, in a first approximation, by their small-signal inductance, L , and their second-order and third-order nonlinearity coefficients, β and γ , respectively. Unless otherwise noted, the analysis in the following sections applies to all inductive elements listed in Tab. 2.1.

Tab. 2.1.: Overview on nonlinear inductive elements based on Josephson junctions.

Element	L	β	γ	Notes
	$\frac{L_{J,0}}{\cos\phi_{dc}}$	$\frac{1}{2} \tan\phi_{dc}$	$\frac{1}{6}$	a
	$\frac{L_{J,0}}{\tilde{r} \cos\phi_{dc} + \mathcal{N}^{-1} \cos\left(\frac{\phi_{dc} - \phi_{ext}}{\mathcal{N}}\right)}$	$\frac{1}{2} \frac{\tilde{r} \sin\phi_{dc} + \mathcal{N}^{-2} \sin\left(\frac{\phi_{dc} - \phi_{ext}}{\mathcal{N}}\right)}{\tilde{r} \cos\phi_{dc} + \mathcal{N}^{-1} \cos\left(\frac{\phi_{dc} - \phi_{ext}}{\mathcal{N}}\right)}$	$\frac{1}{6} \frac{\tilde{r} \cos\phi_{dc} + \mathcal{N}^{-3} \sin\left(\frac{\phi_{dc} - \phi_{ext}}{\mathcal{N}}\right)}{\tilde{r} \cos\phi_{dc} + \mathcal{N}^{-1} \cos\left(\frac{\phi_{dc} - \phi_{ext}}{\mathcal{N}}\right)}$	b
	$\frac{L_m}{1 + \beta_L \cos\phi_{dc}}$	$\frac{\beta_L}{2} \frac{\sin\phi_{dc}}{1 + \beta_L \cos\phi_{dc}}$	$\frac{\beta_L}{6} \frac{\cos\phi_{dc}}{1 + \beta_L \cos\phi_{dc}}$	c
	$\frac{L_m}{L_m \frac{1 + \alpha_p \beta_L \cos\phi_{dc}}{1 + \beta_L \cos\phi_{dc}}}$	$\frac{\beta_L (1 - \alpha_p) \sin\phi_{dc}}{2(1 + \beta_L \cos\phi_{dc})(1 + \alpha_p \beta_L \cos\phi_{dc})^2}$	$\frac{\beta_L (1 - \alpha_p)}{6} \frac{(1 + \alpha_p \beta_L \cos\phi_{dc}) \cos\phi_{dc} + 3\alpha_p \beta_L \sin^2\phi_{dc}}{(1 + \beta_L \cos\phi_{dc})(1 + \alpha_p \beta_L \cos\phi_{dc})^4}$	d
	$\frac{L_m}{L_m \frac{1 + r \cos\phi_{dc}}{1 + \beta_L \cos\phi_{dc}}}$			e
	$\frac{L_f + L_m}{L_f + L_m \frac{1 + \alpha_p \beta_L \cos\phi_{dc}}{1 + \beta_L \cos\phi_{dc}}}$	$\frac{p \beta_L (1 - \alpha_f) \sin\phi_{dc}}{2(1 + \beta_L \cos\phi_{dc})(1 + \alpha_f \beta_L \cos\phi_{dc})^2}$	$\frac{p^2 \beta_L (1 - \alpha_f)}{6} \frac{(1 + \alpha_f \beta_L \cos\phi_{dc}) \cos\phi_{dc} + 3\alpha_f \beta_L \sin^2\phi_{dc}}{(1 + \beta_L \cos\phi_{dc})(1 + \alpha_f \beta_L \cos\phi_{dc})^4}$	f

^a See Sec. 2.1.1. With $L_{J,0} = \Phi_0/2\pi I_c$. In the case of the dc-SQUID, the effective critical current is $I_c^{\text{eff}} = 2I_c |\cos(\Phi_{\text{int}}/2\Phi_0)|$.

^b Valid for $\mathcal{N} > 1$ large junctions. Equations adapted from Ref. [80].

^c With $\beta_L = 2\pi L_m I_c / \Phi_0$. See Sec. 2.1.2.

^d With $\alpha_p = L_p / (L_m + L_p)$ and $\beta_L = 2\pi(L_m + L_p)I_c / \Phi_0$. See Sec. 2.1.3 and Appendix A.1.1.

^e Valid for $r \ll 1$. With $\beta_L = (2\pi L_m I_c / \Phi_0) + r$. Nonlinearity coefficients are the same as in case d, using $L_p = L_{J,0,L} = r\Phi_0/2\pi I_c$. See Sec. 2.1.3 and Appendix A.1.2.

^f With $L_p = L_{p,g} + L_{J,0,L}$, $\alpha_p = L_p / (L_m + L_p)$, $\alpha_f = (\alpha_p L_m + L_p) / (L_m + L_p)$, $p = L_m / (L_m + L_p)$, and $\beta_L = 2\pi(L_m + L_p)I_c / \Phi_0$. See Sec. 2.1.3 and Appendix A.1.3.

2.2 Linear properties of a discrete transmission line

To assemble a discrete traveling-wave metamaterial, we arrange a series array of identical nonlinear inductive elements, described in the preceding section, along with capacitors to ground. This configuration forms a ladder-like transmission line and allows impedance matching to exterior circuitry with a standard characteristic impedance of $Z_0 = 50 \Omega$. The term ‘metamaterial’ reflects that the transmission line consists of elementary unit cells, composed of discrete lumped circuit elements (inductors, capacitors, and resistors), and with physical lengths much smaller than wavelengths at typical operating frequencies. The linear and nonlinear wave-propagation properties of this metamaterial can be engineered on a sub-wavelength scale through the architecture of its unit cells. This capability is extensively utilized in this thesis for the purpose of shaping various parametric processes, both desired and undesired (see Sec. 3.2).

Before we derive the equations of motion describing the nonlinear transmission line (Sec. 2.3), we examine the linear properties of such a transmission line from a microwave-engineering point of view.

2.2.1 Iterative-impedance approach

We start with a transmission line consisting of cascaded identical two-ports with series impedances Z_1 and shunt impedances Z_2 (Fig. 2.10a). It is useful to divide this transmission line into symmetric ‘pi’-type (or Π -type) two-ports with shunt impedances $2Z_2$ each, as shown in Fig. 2.10b. Symmetric ‘T’-type two-ports with two series impedances of $Z_1/2$ each, and shunt impedance Z_2 , yield the same results, but the pi-type topology closely resembles our TWPA implementation (c.f. Fig. 4.2). We assume that the transmission line is terminated by its characteristic impedance, Z_{Π} . Then, the input impedance $Z_{\text{in}} = V_n/I_n$ of any two-port with index n is equal to its load impedance, which itself is equal to the termination impedance

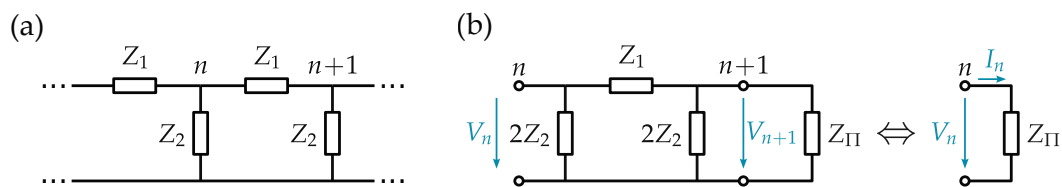


Fig. 2.10.: General discrete transmission line model. **(a)** Infinite transmission line with series impedance Z_1 and shunt impedance Z_2 . **(b)** Transmission line divided into symmetric pi-type unit cells and terminated by its characteristic impedance Z_{Π} . The input impedance of each unit cell is equal to its load impedance, Z_{Π} .

Z_{Π} . This allows calculating Z_{Π} iteratively, as described already by Feynman in his famous lectures on physics [81]. It can be shown that

$$Z_{\Pi} = \sqrt{\frac{Z_1 Z_2}{1 + Z_1/(2Z_2)}}, \quad (2.42)$$

see for example Ref. [82]. Because each unit cell is loaded with the same impedance, no reflections of propagating waves occur, and wave propagation properties can be described by a scalar complex transfer function $H = V_{n+1}/V_n = e^{-\tilde{\gamma}}$. Here, $\tilde{\gamma} = \tilde{\alpha} + j\tilde{\beta}$ denotes the complex propagation constant, where $\tilde{\alpha}$ is the attenuation per unit cell, and $\tilde{\beta}$ is the phase shift per unit cell. Note that if the transmission line is not terminated with Z_{Π} , a more involved approach is necessary. In general, wave propagation is not described by a single scalar function, but instead by a matrix, e.g. the scattering matrix. This matrix can be obtained via the transfer-matrix formalism (Sec. 3.2.3), of which great use is made throughout this thesis. When correctly terminated, a cascade of N unit cells is described by the transfer function $H^N = V_{\text{out}}/V_{\text{in}} = e^{-\tilde{\gamma}N}$. Calculating $e^{-\tilde{\gamma}} = H = V_{n+1}/V_n$ for the two-port shown in Fig. 2.10b yields a lengthy expression, which simplifies to [82]

$$\cosh \tilde{\gamma} = 1 + \frac{Z_1}{2Z_2}. \quad (2.43)$$

2.2.2 Lossless discrete transmission line

Neglecting losses in capacitors and inductive elements for now, the circuit model contains nonlinear inductive elements with inductance L and capacitance C_J , shunted to ground by capacitance C_0 (Fig. 2.11). The capacitance C_J represents the Josephson junction capacitance in the case of a Josephson junction or an ideal rf-SQUID as nonlinear inductive element. For SNAILS or nonideal rf-SQUIDs (Tab. 2.1) the equivalent circuit of a unit cell is more complicated (see Ref. [83] and Appendix A.1.3, respectively), but the general approach described in the following remains the same. The approach follows the modeling of a lossless rf-SQUID-based TWPA by Dixon et al. [58].

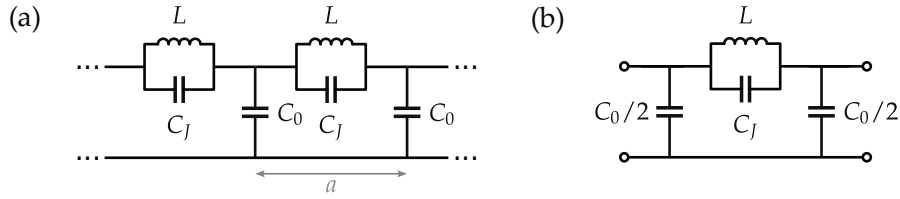


Fig. 2.11.: (a) Transmission line with inductive elements and ground capacitors as basic TWPA architecture. One unit cell has a physical length a . The inductances L represent, e.g., rf-SQUIDs, and C_J is the capacitance of the Josephson junction. (b) Pi-type unit cell.

Using Eq. 2.42, we find the characteristic impedance of the transmission line,

$$\begin{aligned}
 Z(\omega) &= \sqrt{\frac{L}{C_0(1 - \omega^2 LC_J - \omega^2 LC_0/4)}} \\
 &= \sqrt{\frac{L}{C_0(1 - \omega^2/\omega_J^2 - \omega^2/4\omega_0^2)}} \\
 &= \sqrt{\frac{L}{C_0(1 - \omega^2/\omega_c^2)}}, \tag{2.44}
 \end{aligned}$$

introducing the characteristic frequency $\omega_0 = 1/\sqrt{LC_0}$, the plasma frequency $\omega_J = 1/\sqrt{LC_J}$, and the cutoff frequency, which will be defined below in Eq. 2.49. The impedance is real-valued, because the discrete transmission line model consists of reactive elements only. At low frequencies $\omega \ll \omega_J, \omega_0$ the impedance is constant,

$$Z \approx \sqrt{\frac{L}{C_0}}. \tag{2.45}$$

In a transmission line with SQUIDs as inductive elements, having a flux-dependent inductance $L(\phi_{dc})$, the impedance can be tuned in-situ via the flux bias, $Z(\phi_{dc})$. This is particularly useful to achieve impedance-matching of the TWPA to the embedding circuitry with $Z_0 = 50 \Omega$.

Since the term Z_1/Z_2 is also real-valued, we can simplify $\cosh \tilde{\gamma}$ (Eq. 2.43) to $\cosh(\tilde{\alpha} + j\tilde{\beta}) = \cosh \tilde{\alpha} \cos \tilde{\beta}$. The imaginary part, $\text{Im}\{\cosh \tilde{\gamma}\} = \sinh \tilde{\alpha} \sin \tilde{\beta}$, is zero, which is fulfilled if either $\sinh \tilde{\alpha} = 0$ or $\sin \tilde{\beta} = 0$. The former is equivalent to $\tilde{\alpha} = 0$ and $\tilde{\beta} \neq 0$, and describes a passband, where waves can propagate without attenuation, whereas the latter describes a stopband, where waves are attenuated

with $\tilde{\alpha} \neq 0$. A cutoff frequency ω_c marks the transition from passband to stopband. In the passband, the phase shift per unit cell $\tilde{\beta}$ is given by

$$\cos \tilde{\beta} = 1 + \frac{Z_1}{2Z_2}, \quad (2.46)$$

using Eq. 2.43 and $\cosh \tilde{\gamma} = \cos \tilde{\beta}$. The number of unit cells, after which a wave propagating along the transmission line accumulated a total phase shift of 2π , is equivalent to the dimensionless wavelength λ/a . Here, a denotes the physical length of one unit cell. Hence, for the one-dimensional metamaterial discussed here, the phase shift per unit cell is equivalent to the dimensionless wavenumber k normalized to the reverse unit cell length, $k = \tilde{k}a$. The wavenumber is defined as $\tilde{k} = 2\pi/\lambda$, and relates phase velocity and frequency as $\tilde{k} = \omega/v_{\text{ph}}$. In the following, the dimensionless wavenumber k is preferred over $\tilde{k}$, because in a discrete circuit it is more meaningful to state the phase shift per unit cell, than stating the phase shift per meter. With $\tilde{\beta} \equiv k$ we can write $\cosh \tilde{\gamma} = \cos k$ in the passband. Using Eq. 2.46 and the identity $\cos k = 1 - 2\sin^2(k/2)$, we obtain the frequency dependence of k , referred to as the dispersion relation,

$$k(\omega) = 2 \arcsin \left(\frac{\omega}{2\omega_0 \sqrt{1 - \omega^2/\omega_J^2}} \right). \quad (2.47)$$

This is an important result, since the dispersive behaviour of the transmission line has major implications on parametric amplification, as detailed in Sec. 2.4.3. The frequency dependences of k and Z are shown in Fig. 2.12. For low frequencies, $\omega \ll \omega_J, \omega_0$, the transmission line exhibits linear dispersion,

$$k \approx \omega/\omega_0, \quad (2.48)$$

applying the small-angle approximation $\arcsin x \approx x$. Equivalently, the phase velocity is roughly constant, $v_{\text{ph}} \approx a\omega_0 = a/\sqrt{LC_0}$. In many TWPA implementations, parametric amplification operates in this regime. For increasing frequencies, the transmission line behaves increasingly dispersive, with increasing wavenumber, or equivalently, decreasing phase velocity. When the phase shift per unit cell reaches $k \equiv \tilde{\beta} = \pi$, wave propagation is no longer possible. This occurs at the cutoff frequency

$$\omega_c = \frac{2\omega_0}{\sqrt{1 + 4\omega_0^2/\omega_J^2}} \quad (2.49)$$

This cutoff is a consequence of the discreteness of the transmission line, which acts as a low-pass filter above ω_c . In contrast, a continuous transmission line, as

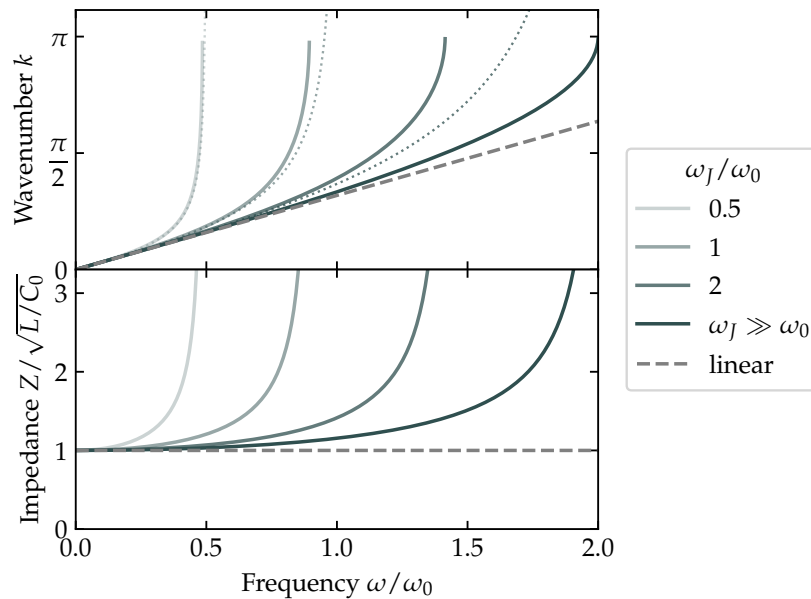


Fig. 2.12.: Frequency dependence of the wavenumber and the characteristic impedance of a lossless discrete transmission line. In the low-frequency limit, the relationships simplify to $k \approx \omega / \omega_0$ and $Z \approx \sqrt{L/C_0}$. Above the cutoff frequency, where $k = \pi$, wave propagation is not possible. Dotted lines depict the continuum approximation, Eq. 2.69, not exhibiting a cutoff. Dashed lines depict a transmission line in the continuum limit with linear dispersion.

described by standard transmission-line theory with inductance L' and capacitance C' per infinitesimal unit length $a \rightarrow 0$, does not possess a cutoff frequency, and allows waves of any frequency to propagate.

2.2.3 Discrete transmission line in the low-loss limit

Transmission-line loss significantly contributes to why TWPAs fail to reach the quantum limit of noise (see Sec. 4.6.6), so understanding its origin and modeling it is crucial for accurately predicting the performance of TWPAs. A realistic Josephson junction, typically modeled with the resistively-capacitively shunted Josephson junction model (RCSJ, see Fig. 2.1), contains a parallel resistor R_J , besides the junction capacitance C_J [78]. Moreover, superconducting inductors exhibit a complex impedance, including a resistive contribution called the surface resistance R_s . Both R_J and R_s are consequences of quasiparticles in the superconductor, explained in more detail in Appendices A.2.1 and A.2.2. Similarly, realistic capacitors can be substantially lossy at microwave frequencies and millikelvin temperatures due to dielectric losses [84]. This loss is usually given via the loss tangent $\tan \delta_C$ and results in a complex impedance of the capacitor, e.g., $Z_{C_0} = 1/(j\omega C_0(1 - j \tan \delta_C))$ in the

case of a ground capacitor C_0 . It can be modeled as a resistor R_C in parallel to C_0 (Fig. 2.13).

It is useful to combine both quasiparticle resistances of the inductive element as an effective resistance R_L in series to the inductance L (Fig. 2.13), as it is commonly done in standard transmission-line theory, where the conductor loss is modeled as a resistance R' in series to the inductance L' . The quasiparticle loss exhibits a roughly quadratic frequency dependence, $R_L \propto \omega^2$, in the relevant frequency range of our TWPA implementation, see Sec. A.2.2, assuming that R_J is constant in frequency. Equivalently, the loss of the inductive element can be modeled as a resistance $R_{\text{sg,eff}} = (\omega L)^2 / R_L$ in parallel to the inductance L , which is done in Sec. 4.2.1 due to its simple implementation via the subgap resistance of the built-in RCSJ model of the WRspice circuit simulator. This effective parallel resistance is nearly frequency-independent in the relevant frequency range.

Here, we consider the behaviour of the discrete transmission line in the low-loss limit, $\tan \delta_C \ll 1$ and $R_L \ll \omega L$. In this regime, the impedance Z and wavenumber k are still well represented by Eqs. 2.44 and 2.47, and the main consequence of the presence of loss is attenuation of the propagating waves. In this limit, $\tilde{\alpha} \ll \tilde{\beta}$, so the attenuation constant can be obtained perturbatively using Eq. 2.43,

$$\cosh \tilde{\gamma} = \cosh \tilde{\alpha} \cos \tilde{\beta} + j \sinh \tilde{\alpha} \sin \tilde{\beta} = 1 + \frac{\tilde{Z}_1}{2\tilde{Z}_2}. \quad (2.50)$$

The complex series and shunt impedances can be expressed via their lossless counterparts as

$$\tilde{Z}_1 = Z_1(1 - j\delta_L), \quad (2.51)$$

$$\tilde{Z}_2 = Z_2/(1 - j\delta_C), \quad (2.52)$$

where $\delta_C \approx \tan \delta_C \ll 1$ and δ_L denote the loss rates of the capacitor and inductor, respectively. The imaginary part of Eq. 2.50 can be simplified using $\sinh \tilde{\alpha} \approx \tilde{\alpha}$ and the identity $\sin \tilde{\beta} = \sqrt{1 - \cos^2 \tilde{\beta}}$. By taking the imaginary part of Eq. 2.50,

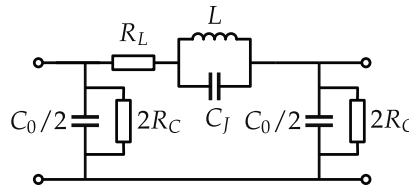


Fig. 2.13.: Schematic of a unit cell of a lossy transmission line. Dielectric loss is represented by $R_C(\omega)$, and quasiparticle loss of the conductor by $R_L(\omega)$.

and expressing β via its lossless formulation, Eq. 2.46, we obtain the attenuation constant of the discrete lossy transmission line,

$$\tilde{\alpha} = (\delta_C + \delta_L) \frac{1}{\sqrt{-4Z_2/Z_1 - 1}}, \quad (2.53)$$

note that $Z_2/Z_1 < 0$. According to the schematic in Fig. 2.13, the loss rates are

$$\delta_L = \frac{R_L}{\omega L} (1 - \omega^2 LC_J), \quad (2.54)$$

$$\delta_C = \frac{1}{\omega C_0 R_C}, \quad (2.55)$$

so the attenuation constant reads

$$\tilde{\alpha} = (\delta_C + \delta_L) \frac{\omega}{2\omega_0} \frac{1}{\sqrt{1 - \omega^2/\omega_c^2}}. \quad (2.56)$$

For low frequencies, $\omega \ll \omega_c, \omega_0, \omega_J$, this equation can be approximated to

$$\tilde{\alpha} \approx \frac{1}{2} \left(R_L \sqrt{\frac{C_0}{L}} + \frac{1}{R_C} \sqrt{\frac{L}{C_0}} \right), \quad (2.57)$$

equivalent to the result from standard (continuous) transmission-line theory [85]. The full expression is particularly important if $\omega_0 \ll \omega_J$, which is the case in the 3WM TWPA concept proposed in Ref. [86], where the pump frequency is placed close to ω_c . In this regime, the attenuation deviates greatly from the continuum approximation. A wave propagating through a transmission line consisting of N unit cells has an output power $P_{\text{out}} = P_{\text{in}} e^{2\tilde{\alpha}N}$, so the attenuation is given as [85]

$$\begin{aligned} A^{\text{dB}} &= 10 \log_{10} e^{2\tilde{\alpha}N} = 20 \tilde{\alpha} N \log_{10} e \\ &\approx 8.686 \tilde{\alpha} N \text{ dB}. \end{aligned} \quad (2.58)$$

The frequency dependence of the attenuation constant is illustrated in Fig. 2.14 for typical values at 4.2 K, where both quasiparticle loss and dielectric loss are substantial (c.f. Tab. 4.4). At low frequencies, dielectric loss increases linearly with frequency, since $R_C = 1/\omega C_0 \tan \delta_C$, whereas quasiparticle loss increases approximately quadratically, $R_L = \omega^2 L / R_{\text{sg,eff}}$. At high frequencies, the attenuation constant increases drastically, approaching the cutoff at ω_c . Due to the exponential temperature dependence of quasiparticle densities in superconductors (Appendices A.2.1 and A.2.2), quasiparticle losses play practically no role at millikelvin temperatures, and the loss in the TWPA is dominated by dielectric loss.

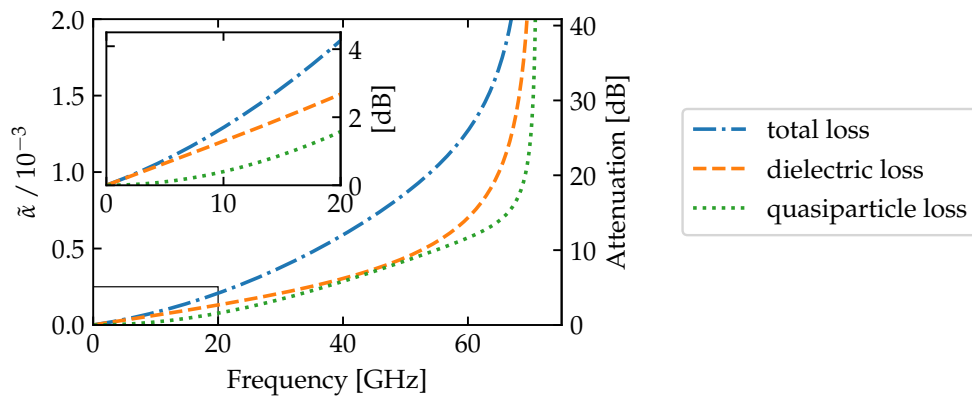


Fig. 2.14.: Attenuation in a superconducting transmission line. At low frequencies (inset) the dielectric loss increases linearly with frequency and the quasiparticle loss quadratically. At high frequencies approaching the cutoff frequency ω_c , the loss increases drastically. Parameters are $L = 100$ pH, $\omega_I = \omega_0 = 2\pi \times 80$ GHz, $\sqrt{L/C_0} = 50 \Omega$, $\tan \delta_C = 10^{-3}$, and $R_{\text{sg,eff}} = 20$ k Ω , representing realistic values at 4.2 K. The attenuation is calculated for a TWPA with $N = 2352$ unit cells.

2.3 Nonlinear discrete transmission line

2.3.1 Discrete nonlinear telegrapher's equation

In this section, we first derive the discrete nonlinear wave equation, often referred to as the telegrapher's equation, for the transmission line. Taking the continuum limit of this equation, valid for low frequencies, allows the derivation of the coupled-mode equations in the next section, which describe the properties arising from the nonlinear coupling of multiple copropagating waves. The derivation broadly follows that for a 4WM TWPA by Yaakobi et al. [87].

The schematic of the discrete transmission line is presented in Fig. 2.15. The transmission line is considered lossless, consisting of reactances only. Dissipative effects are discussed in Sec. 2.2.3 and are not included in the nonlinear modeling. We start with recalling that the voltage on the node with index n relates to the superconducting phase as

$$v_n(t) = \varphi_0 \frac{d\phi_n(t)}{dt}. \quad (2.59)$$

For convenience, the time dependence is not written explicitly in the following. The voltage across two adjacent nodes, $v = v_n - v_{n+1}$, determines the phase difference $\phi = \phi_n - \phi_{n+1}$ across the nonlinear inductive element, as defined in Sec. 2.1. The current flowing through the inductive element is therefore

$$\begin{aligned} i_{L,n} &\approx \frac{\varphi_0}{L} [\phi - \beta\phi^2 - \gamma\phi^3] \\ &= \frac{\varphi_0}{L} [\phi_n - \phi_{n+1} - \beta(\phi_n - \phi_{n+1})^2 - \gamma(\phi_n - \phi_{n+1})^3], \end{aligned} \quad (2.60)$$

see e.g. Eq. 2.28 in the case of an rf-SQUID as nonlinear element. At the same time, this phase difference causes a displacement current in the capacitance C_J ,

$$i_{C_J,n} = C_J \frac{dv}{dt} = \varphi_0 C_J \frac{d^2\phi}{dt^2} = \varphi_0 C_J \ddot{\phi}. \quad (2.61)$$

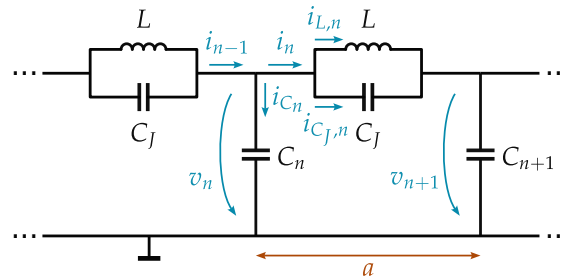


Fig. 2.15.: Transmission with nonlinear inductive elements and capacitors to ground as basic TWPA architecture. The inductances L represent, e.g., rf-SQUIDs.

which, due to current conservation, $i_n = i_{L,n} + i_{C_J,n}$, yields

$$i_n = \varphi_0 C_J (\ddot{\phi}_n - \ddot{\phi}_{n+1}) + \frac{\varphi_0}{L} [\phi_n - \phi_{n+1} - \beta(\phi_n - \phi_{n+1})^2 - \gamma(\phi_n - \phi_{n+1})^3]. \quad (2.62)$$

This current relates to the current through the preceding unit cell as

$$i_{n-1} - i_n = i_{C_n} = \varphi_0 C_n \ddot{\phi}_n \quad (2.63)$$

due to current conservation in node n . Inserting Eq. 2.62 into Eq. 2.63 and rearranging leads to a set of equations,

$$(\phi_{n+1} - 2\phi_n + \phi_{n-1}) - LC_n \ddot{\phi}_n + LC_J (\ddot{\phi}_{n+1} - 2\ddot{\phi}_n + \ddot{\phi}_{n-1}) + \beta [(\phi_{n+1} - \phi_n)^2 - (\phi_n - \phi_{n-1})^2] - \gamma [(\phi_{n+1} - \phi_n)^3 - (\phi_n - \phi_{n-1})^3] = 0, \quad (2.64)$$

to which we refer as the discrete nonlinear telegrapher's equation. This set of equations is similar to what is solved by a time-domain circuit simulator such as SPICE. These equations could also be solved directly, either analytically, or by numerical means. The former has been done for the linearized set of equations in Ref. [83], yielding e.g. the dispersion relation, Eq. 2.47. The latter has been carried out with a plane-wave ansatz for the waves of pump, signal, and idler to describe parametric amplification in Ref. [50]. Alternatively, this discrete set of differential equations can be approximated by the continuous nonlinear telegrapher's equation, a single differential equation in the continuous limit [87]. Further approximations lead to the coupled-mode equations, which allow analytic solutions for several 3WM and 4WM parametric processes. To obtain analytical expressions we take the continuous-limit approach (Sec. 2.4), with the price of reduced accuracy due to the various applied approximations. Later, we improve on the accuracy of the analytical solutions by applying SPICE circuit simulations (Chapter 3).

2.3.2 Continuous nonlinear telegrapher's equation

For the transition from the discrete to the continuous nonlinear telegrapher's equation, we consider now a homogeneous transmission line with identical ground capacitors $C_n = C_0$. In the case of periodically varied ground capacitors, which is one of the core techniques for dispersion engineering in this thesis (Chapters 3 and 4), C_0 corresponds to the mean value of the ground capacitors, $\overline{C}$.

We approximate the phase $\phi_n(t)$ at a discrete node n by a continuous function of the phase $\phi(\tilde{x}, t)$ at a continuous position $\tilde{x}$. This approximation is adequate for low frequencies, for which the corresponding wavelengths are much larger than the physical length of one unit cell, $\lambda \gg a$. To make the position coordinate

dimensionless, we set $x = \tilde{x}/a$. With these definitions and assumptions, the finite differences appearing in Eq. 2.64 can be approximated by a continuous derivative,

$$\phi_{n+1} - 2\phi_n + \phi_{n-1} \approx \frac{\partial^2 \phi}{\partial x^2}. \quad (2.65)$$

Adapting the approach of Ref. [87], the nonlinear terms can be approximated as

$$(\phi_{n+1} - \phi_n)^2 - (\phi_n - \phi_{n-1})^2 \approx \frac{\partial}{\partial x} \left[\left(\frac{\partial \phi}{\partial x} \right)^2 \right], \quad (2.66)$$

$$(\phi_{n+1} - \phi_n)^3 - (\phi_n - \phi_{n-1})^3 \approx \frac{\partial}{\partial x} \left[\left(\frac{\partial \phi}{\partial x} \right)^3 \right], \quad (2.67)$$

resulting in the continuous nonlinear telegrapher's equation [57],

$$\frac{\partial^2 \phi}{\partial x^2} - \omega_0^{-2} \frac{\partial^2 \phi}{\partial t^2} + \omega_J^{-2} \frac{\partial^4 \phi}{\partial t^2 \partial x^2} = -\beta \frac{\partial}{\partial x} \left[\left(\frac{\partial \phi}{\partial x} \right)^2 \right] + \gamma \frac{\partial}{\partial x} \left[\left(\frac{\partial \phi}{\partial x} \right)^3 \right]. \quad (2.68)$$

This equation describes the evolution and propagation of waves in the nonlinear transmission line in the long-wavelength approximation. The nonlinear terms are written to the right hand side of the equation, because they can be interpreted as source terms in the case of coupled waves, as will become apparent later.

Omitting the nonlinear (source) terms for now, the resulting linear homogeneous wave equation can be solved by a plane-wave ansatz $\phi(x, t) = Ae^{j(kx - \omega t)}$, with a constant amplitude A . The solution is the dimensionless wavenumber

$$k(\omega) = \pm \frac{\omega}{\omega_0 \sqrt{1 - \omega^2/\omega_J^2}}, \quad (2.69)$$

where the sign '+' corresponds to a wave propagating from left to right, and vice versa for '-'. This equation can be understood as the small-phase-shift approximation of the more general dispersion relation obtained via the iterative-impedance approach, Eq. 2.47. For $\omega_J < \omega_0$, Eq. 2.69 is reasonably close to the full expression, see Fig. 2.12. The main difference, however, is the absence of a cutoff, above which waves cannot propagate.

Next, we allow spatially slowly varying amplitudes $A(x)$ of waves, satisfying

$$\left| \frac{d^2 A}{dx^2} \right| \ll k \left| \frac{dA}{dx} \right| \ll k^2 |A|, \quad (2.70)$$

called the slowly-varying envelope approximation [59]. As simulations of a typical circuit without above approximations show, the latter approximation is well

justified at least for the main three waves involved in parametric amplification (refer to Fig. 3.5). Further, their amplitudes may be complex, $A = |A|e^{j\chi}$, including initial phases χ , and A^* denoting the complex conjugate (c.c.), $AA^* = |A|^2$. After inserting

$$\phi(x, t) = \frac{1}{2} [A(x)e^{j(kx-\omega t)} + A^*(x)e^{-j(kx-\omega t)}], \quad (2.71)$$

into Eq. 2.68, all terms proportional to A, A^* cancel out due to the plane-wave solution, Eq. 2.69. After neglecting second derivatives (Eq. 2.70), but keeping first derivatives, the left hand side of the nonlinear telegrapher's equation simplifies for small frequencies, $\omega \ll \omega_J$, to

$$\frac{\partial^2 \phi}{\partial x^2} - \omega_0^{-2} \frac{\partial^2 \phi}{\partial t^2} + \omega_J^{-2} \frac{\partial^4 \phi}{\partial t^2 \partial x^2} = jk \left[\frac{dA}{dx} e^{j(kx-\omega t)} - \frac{dA^*}{dx} e^{-j(kx-\omega t)} \right] \quad (2.72)$$

describing the evolution of amplitudes for given nonlinear source terms, the right hand side of Eq. 2.68.

2.4 Coupled-mode theory

Parametric processes between several copropagating waves in nonlinear media, such as nonlinear optical fibers or TWPAs, can be described via the coupled-mode equations (CMEs) [56]. This approach allows analytical solutions of the wave equation for a number of parametric processes in both 3WM and 4WM, including but not limited to parametric amplification, frequency conversion, and second-harmonic and third-harmonic generation. The following presentation is partially based on the derivations and findings of Zorin [57].

2.4.1 Coupled-mode equations

We start from a three-mode ansatz

$$\phi(x, t) = \frac{1}{2} \left[A_p e^{j(k_p x - \omega_p t)} + A_s e^{j(k_s x - \omega_s t)} + A_i e^{j(k_i x - \omega_i t)} + A_p^* e^{-j(k_p x - \omega_p t)} + A_s^* e^{-j(k_s x - \omega_s t)} + A_i^* e^{-j(k_i x - \omega_i t)} \right], \quad (2.73)$$

where indices p, s, i denote pump, signal, and idler, relating to each other via $\omega_p = \omega_s + \omega_i$ in 3WM, satisfying energy conservation. Corresponding wavenumbers are given by Eq. 2.69. It is essential to include the complex conjugates, often omitted in the notation of linear equations. Inserting the three-mode ansatz into Eq. 2.68, and applying again the slowly-varying envelope approximation, Eq. 2.70, the quadratic

mode-coupling term on the right hand side of the differential equation contains $6^2 = 36$ terms,

$$\left(\frac{\partial \phi}{\partial x}\right)^2 = -\frac{1}{4} \left[\begin{aligned} &k_p^2 A_p^2 e^{j(2k_p x - 2\omega_p t)} + \text{c.c.} && \text{SHG} && \omega_{2p} = 2\omega_p \\ &+ k_s^2 A_s^2 e^{j(2k_s x - 2\omega_s t)} + \text{c.c.} && \text{SHG} && \omega_{2s} = 2\omega_s \\ &+ k_i^2 A_i^2 e^{j(2k_i x - 2\omega_i t)} + \text{c.c.} && \text{SHG} && \omega_{2i} = 2\omega_i \\ &- 2k_p^2 |A_p|^2 - 2k_s^2 |A_s|^2 - 2k_i^2 |A_i|^2 && \text{Rect.} && \omega = 0 \\ &+ 2k_p k_s A_p A_s e^{j((k_p + k_s)x - (\omega_p + \omega_s)t)} + \text{c.c.} && \text{SFG} && \omega_{p+s} = \omega_p + \omega_s \\ &- 2k_p k_s A_p A_s^* e^{j((k_p - k_s)x - (\omega_p - \omega_s)t)} - \text{c.c.} && \text{DFG} && \omega_i = \omega_p - \omega_s \\ &+ 2k_p k_i A_p A_i e^{j((k_p + k_i)x - (\omega_p + \omega_i)t)} + \text{c.c.} && \text{SFG} && \omega_{p+i} = \omega_p + \omega_i \\ &- 2k_p k_i A_p A_i^* e^{j((k_p - k_i)x - (\omega_p - \omega_i)t)} - \text{c.c.} && \text{DFG} && \omega_s = \omega_p - \omega_i \\ &+ 2k_s k_i A_s A_i e^{j((k_s + k_i)x - (\omega_s + \omega_i)t)} + \text{c.c.} && \text{SFG} && \omega_p = \omega_s + \omega_i \\ &- 2k_s k_i A_s A_i^* e^{j((k_s - k_i)x - (\omega_s - \omega_i)t)} - \text{c.c.} && \text{DFG} && \omega_{s-i} = \omega_s - \omega_i \end{aligned} \right], \quad (2.74)$$

representing the 3WM parametric processes of second-harmonic generation (SHG), sum-frequency generation (SFG), also called up-conversion, difference-frequency generation (DFG), also called down-conversion, and rectification (Rect., c.f. Sec. 2.1.2). Similarly, the third-order mode-coupling term contains $6^3 = 216$ terms,

$$\left(\frac{\partial \phi}{\partial x}\right)^3 = -j\frac{1}{8} \left[\begin{aligned} &k_p^3 A_p^3 e^{j(3k_p x - 3\omega_p t)} + \text{c.c.} && \text{THG} && \omega_{3p} = 3\omega_p \\ &- 3k_p^3 A_p |A_p|^2 e^{j(k_p x - \omega_p t)} + \text{c.c.} && \text{SPM} && \omega_p = 2\omega_p - \omega_p \\ &- 6k_p^2 k_s |A_p|^2 A_s e^{j(k_s x - \omega_s t)} + \text{c.c.} && \text{XPM} && \omega_s = \omega_s + \omega_p - \omega_p \\ &- 6k_p^2 k_i |A_p|^2 A_i e^{j(k_i x - \omega_i t)} + \text{c.c.} && \text{XPM} && \omega_i = \omega_i + \omega_p - \omega_p \\ &- 3k_p^2 k_s A_p^2 A_s^* e^{j((2k_p - k_s)x - (2\omega_p - \omega_s)t)} + \text{c.c.} && \text{DFG} && \omega_{i,1} = 2\omega_p - \omega_s = \omega_{p+i} \\ &- 3k_p^2 k_i A_p^2 A_i^* e^{j((2k_p - k_i)x - (2\omega_p - \omega_i)t)} + \text{c.c.} && \text{DFG} && \omega_{i,2} = 2\omega_p - \omega_i = \omega_{p+s} \\ &\pm \dots \end{aligned} \right], \quad (2.75)$$

and many other terms producing frequencies higher than $2\omega_p$, or involving higher powers of the signal and idler amplitudes. The terms of Eq. 2.75 represent degen-

erate 4WM parametric processes, called degenerate because both pump waves share the same frequency, $f_{p,1} = f_{p,2} = f_p$. The 4WM processes include third-harmonic generation (THG), self-phase modulation (SPM), cross-phase modulation (XPM), difference-frequency generation (DFG), and sum-frequency generation (SFG, not shown). Parametric amplifiers based on 4WM typically utilize the process $2\omega_p = \omega_s + \omega_{i,1}$, here defined as degenerate DFG, as the idler is a difference frequency. For another illustrative explanation of SPM in rf-SQUID-based metamaterials see Sec. 2.1.2. While six terms constitute SPM, twelve terms constitute the XPMs of the signal and the idler each, explaining why the XPM is twice stronger in effect than SPM.

For the further derivations, we keep only the terms producing the frequencies $\omega_p, \omega_s, \omega_i$. Utilized for 3WM parametric amplification are the two DFG processes involving ω_s and ω_i , whereas all other processes are parasitic in the context of parametric amplification. For illustration, a simulated output spectrum of a TWPA with typical circuit parameters (but without dispersion engineering) is presented in Fig. 2.16, suggesting that some tones like the SHG of signal and idler, can be safely neglected, while others, including SHG of the pump, and SFG producing the tones f_{p+s} and f_{p+i} , can have similar amplitudes as the main tones, i.e., the pump, and the signal and idler, respectively. Although the following three-mode CMEs predict the generation of other modes, they do not cover their impact on the system dynamics. For circuits with low dispersion, this obstacle can severely limit the accuracy of the CME theory, as demonstrated by Dixon et al. [58], an issue further discussed below.

In general, the wavenumbers of the three main tones do not satisfy the phase-matching condition $k_p = k_s + k_i$, but have a phase mismatch

$$\Delta k = k_p - k_s - k_i, \quad (2.76)$$

given for example by the dispersion relation in the continuous limit, Eq. 2.69, or by dispersion relations of more complex circuits, as those discussed in Sec. 3.2. Substituting, in the example of the idler wave, $k_p - k_s = k_i + \Delta k$ and $\omega_p - \omega_s = \omega_i$ into Eq. 2.74, its derivative reads

$$\frac{\partial}{\partial x} \left[\left(\frac{\partial \phi}{\partial x} \right)^2 \right] = j \frac{1}{2} k_i k_p k_s \left[A_p A_s^* e^{j\Delta k x} e^{j(k_i x - \omega_i t)} - \text{c.c.} \right] \pm \dots, \quad (2.77)$$

and similarly for the signal and pump waves. Here, the slowly-varying envelope approximation, Eq. 2.70, was employed once more, and Δk was neglected in the coefficient ($\Delta k \ll 1$), but not in the exponent. In a similar fashion, the derivative of Eq. 2.74 is taken. Then, these nonlinear terms and the result of the homogeneous

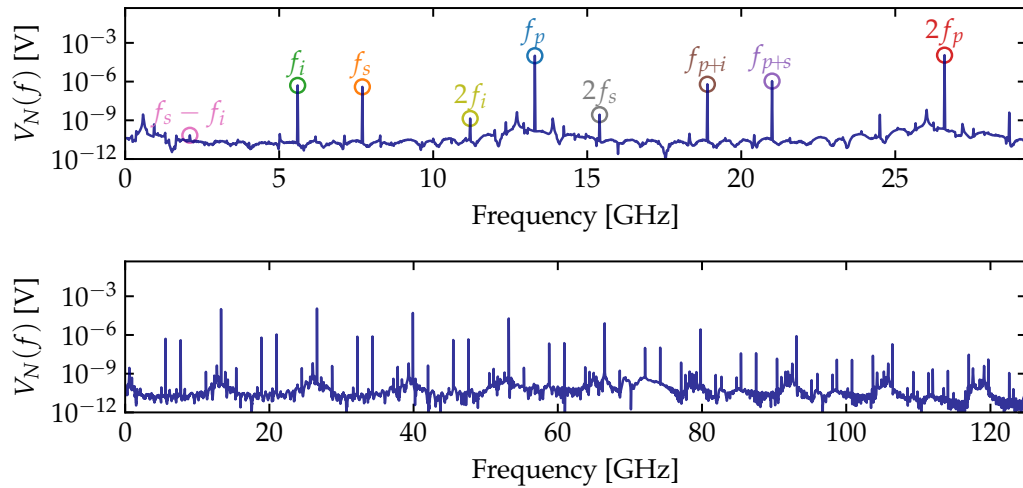


Fig. 2.16.: Simulated frequency spectrum of the output node voltage of a TWPA. Various 3WM products are annotated. The broadband nature of the circuit facilitates a multitude of harmonics and higher-order mixing products, far beyond the three-mode model. Data is obtained via WRspice circuit simulation, see Chapter 3. Parameters of the circuit are similar to those of Fig. 3.5a, but at a signal frequency of 7.7 GHz.

equation, Eq. 2.72, can be inserted into the nonlinear telegrapher's equation, Eq. 2.68. After separating it into three coupled equations, one for each frequency $\omega_p, \omega_s, \omega_i$, we arrive at the coupled-mode equations [57]

$$\frac{dA_p}{dx} = j\frac{3}{8}\gamma k_p^3 |A_p|^2 A_p + \frac{\beta}{2} k_s k_i A_s A_i e^{-j\Delta kx}, \quad (2.78)$$

$$\frac{dA_s}{dx} = j\frac{3}{4}\gamma k_s k_p^2 |A_p|^2 A_s - \frac{\beta}{2} k_i k_p A_p A_i^* e^{j\Delta kx}, \quad (2.79)$$

$$\frac{dA_i}{dx} = j\frac{3}{4}\gamma k_i k_p^2 |A_p|^2 A_i - \frac{\beta}{2} k_s k_p A_p A_s^* e^{j\Delta kx}, \quad (2.80)$$

The terms proportional to β represent parametric amplification. By these terms the equations are coupled to each other. Ignoring the terms proportional to γ , the right hand side of the equation for the mode with index $m \in \{p, s, i\}$ does not contain terms proportional to the amplitude A_m . This is why these nonlinear terms can be understood as source terms. The terms proportional to $\gamma |A_p|^2$ represent SPM and XPM, driven by a strong pump, and mediated by the Kerr nonlinearity with coefficient γ [56].

2.4.2 Self-phase and cross-phase modulations

In the following derivations, we consider a strong pump wave with constant amplitude $|A_p| = A_{p0} \gg |A_s|, |A_i|$. Deviations from this so called undepleted-pump approximation (sometimes called stiff-pump approximation), i.e., signal and idler amplitudes approaching that of the pump, and the resulting saturation effects, are discussed in Sec. 3.4. Under this condition, the pump equation, Eq. 2.78, is effectively decoupled from the signal and idler equations, and reads

$$\frac{dA_p}{dx} = j\frac{3}{8}\gamma k_p^3 A_{p0}^2 A_p. \quad (2.81)$$

A solution to this equation is a complex pump amplitude of the form $A_p(x) = A_{p0}e^{j\vartheta_p x}$, setting the initial phase at $x = 0$ to $\chi_{p0} = 0$ without losing generality. Inserting this ansatz into Eq. 2.81 yields

$$\vartheta_p = \frac{3}{8}\gamma k_p^3 A_{p0}^2, \quad (2.82)$$

and, when setting $|A_s| = |A_i| = 0$, the resulting node-phase wave is

$$\phi(x, t) = \frac{1}{2} \left[A_{p0} e^{j((k_p + \vartheta_p)x - \omega_p t)} + \text{c.c.} \right]. \quad (2.83)$$

Therefore, ϑ_p represents a nonlinear ($\propto A_{p0}^2$) phase shift of the pump wave, additive to the wavenumber. It is called self-phase modulation (SPM), because the pump wave modulates its own propagation properties [56]. In a similar fashion, the terms $\propto |A_p|^2$ in the signal and idler equations, Eqs. 2.79 and 2.80, correspond to the cross-phase modulations (XPM),

$$\vartheta_s = \frac{3}{4}\gamma k_p^2 k_s A_{p0}^2, \quad (2.84)$$

$$\vartheta_i = \frac{3}{4}\gamma k_p^2 k_i A_{p0}^2. \quad (2.85)$$

These nonlinear phase shifts constitute the nonlinear phase mismatch Δk_{NL} for parametric amplification, resulting in a total phase mismatch

$$\Delta k_{\text{tot}} = k_p - k_s - k_i + \vartheta_p - \vartheta_s - \vartheta_i. \quad (2.86)$$

This constitutes one of the main obstacles in 4WM TWPAs, where the SPM and XPM effects are not separated from parametric amplification, because they are mediated by the same Kerr nonlinearity. This fact requires dedicated phase-matching techniques to achieve exponential signal amplification [59]. In 3WM TWPAs, however, these effects are separated, because they are proportional to different nonlinearity

coefficients: parametric amplification to β , and SPM and XPM to γ , which can be much smaller than β . Moreover, in rf-SQUID-based and SNAIL-based TWPAs it is possible to achieve $\gamma = 0$, eliminating SPM and XPM altogether with the nonlinear phase mismatch [57]. In the following, we treat the Kerr-free case, $\gamma = 0$. Nevertheless, the results obtained in the next section remain valid even for $\gamma \neq 0$, when substituting Δk by Δk_{tot} .

2.4.3 Parametric amplification in a Kerr-free TWPA under phase-matched conditions

Under the conditions of a strong, undepleted pump, and $\gamma = 0$, the signal and idler CMEs, Eqs. 2.79 and 2.80, simplify to

$$\frac{dA_s}{dx} = -\frac{\beta}{2}k_i k_p A_{p0} A_i^* e^{j\Delta k x}, \quad (2.87)$$

$$\frac{dA_i}{dx} = -\frac{\beta}{2}k_s k_p A_{p0} A_s^* e^{j\Delta k x}. \quad (2.88)$$

After neglecting the phase mismatch Δk for now, differentiating Eq. 2.87, and inserting Eq. 2.88, we get a single differential equation for $A_s(x)$,

$$\begin{aligned} \frac{d^2 A_s}{dx^2} &= -\frac{\beta}{2}k_i k_p A_{p0} \frac{dA_i^*}{dx} \\ &= \frac{\beta^2}{4}k_s k_i k_p^2 A_{p0}^2 A_s \\ &= g^2 A_s, \end{aligned} \quad (2.89)$$

introducing $g = (1/2)\beta k_p A_{p0} \sqrt{k_s k_i}$. A solution to this equation is

$$A_s(x) = A_{s,1} \sinh gx + A_{s,2} \cosh gx, \quad (2.90)$$

with coefficients determined by the boundary conditions. At the TWPA input, the complex signal amplitude is $A_s(x=0) = A_{s0} e^{j\chi_{s0}}$, with a real-valued amplitude A_{s0} and an initial phase χ_{s0} relative to that of the pump, which is set to $\chi_{p0} = 0$ without losing generality. No idler wave is injected into the TWPA, $A_i(x=0) = 0$. These boundary conditions require $A_{s,1} = 0$ and $A_{s,2} = A_{s0} e^{j\chi_{s0}}$. The complex idler amplitude $A_i(x)$ can be found via differentiating Eq. 2.90 and inserting it to Eq. 2.87, leading to

$$A_i^*(x) = -A_{s0} e^{j\chi_{s0}} \sqrt{\frac{k_s}{k_i}} \sinh gx. \quad (2.91)$$

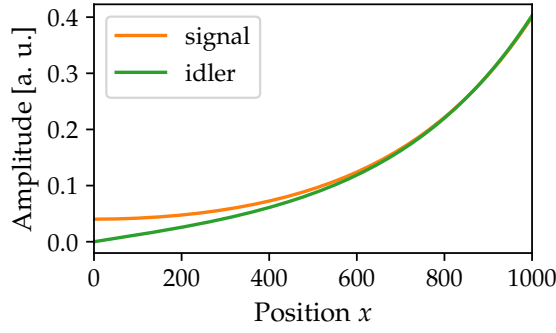


Fig. 2.17.: Signal and idler amplitudes under phase-matched conditions. The amplitudes increase exponentially while the waves propagate along the circuit, Eqs. 2.92 and 2.93.

Using the identity $-1 = e^{j\pi}$, and taking the complex-conjugate of Eq. 2.91, the initial phase of $A_i(x=0)$ is $\pi - \chi_{s0}$. With this, the complex amplitudes of the signal and idler waves are

$$A_s(x) = A_{s0} e^{j\chi_{s0}} \cosh gx, \quad (2.92)$$

$$A_i(x) = A_{s0} e^{j(\pi - \chi_{s0})} \sqrt{\frac{k_s}{k_i}} \sinh gx, \quad (2.93)$$

showing exponential increase of amplitudes when propagating along the circuit (Fig. 2.17). Due to small dispersion, $k \approx \omega/\omega_0$ (neglected phase mismatch), and power $P_i \propto |A_i|^2$, we can interpret the coefficient $k_s/k_i \approx \omega_s/\omega_i$ as a normalization factor accounting for the different photon energies of signal and idler, $\hbar\omega_s$ and $\hbar\omega_i$ (c.f. the Manley-Rowe relations, Sec. 3.4.2). Parametric amplification utilizes the down-conversion process, which can be compared to parametric up-conversion (frequency conversion), derived in Sec. 4.4, where the roles of signal and pump are inverted, such that the wave with the lower frequency ('signal') has the larger amplitude. In this case, the solution is of the form of $\sin \delta x$ and $\cos \delta x$, instead of $\sinh gx$ and $\cosh gx$ for parametric amplification.

We find the signal power gain as the squared ratio of the signal amplitude at the TWPA output, $x = N$, and that at the input, $x = 0$,

$$G = \left| \frac{A_s(N)}{A_s(0)} \right|^2 = \cosh^2(gN). \quad (2.94)$$

Consequently, in the absence of phase mismatch, the achievable gain of a TWPA increases exponentially with device length. Furthermore, this gain does not depend on the initial phases. Parametric amplification is phase-insensitive, or phase-

preserving, with the exception at $f_s \neq f_i$, discussed below. To investigate the signal frequency dependence of the signal power gain, we rewrite

$$g = \frac{1}{2} \beta k_p A_{p0} \sqrt{k_s k_i} = g_0 \sqrt{1 - \delta^2}, \quad (2.95)$$

where g_0 denotes the exponential gain factor in the center of the gain band,

$$g_0 = \frac{1}{4} |\beta| A_{p0} k_p^2, \quad (2.96)$$

Using $k \approx \omega / \omega_0$, valid for $\omega \ll \omega_c$, we define δ as the dimensionless detuning of the signal and idler frequencies from the center of the gain band at $f_p/2$,

$$\delta = \frac{|f_s - f_p/2|}{f_p/2} = \frac{|f_i - f_p/2|}{f_p/2}. \quad (2.97)$$

Substituting g in Eq. 2.94, the frequency dependence of the gain is

$$G = \cosh^2 \left(g_0 \sqrt{1 - \delta^2} N \right), \quad (2.98)$$

exhibiting a semicircle shape due to $\propto \sqrt{1 - \delta^2}$ when plotted logarithmically, e.g. in decibels, $G^{\text{dB}} = 10 \log_{10} G$ dB (Fig. 2.18).

Furthermore, similar to Eq. 2.94, a signal-to-idler conversion gain (or intermodulation gain) can be calculated, which is, after normalizing by the corresponding photon energies, $\hbar\omega_s$ and $\hbar\omega_i$,

$$G_{si} = \left| \frac{A_i(N)}{A_s(0)} \right|^2 \frac{\omega_s}{\omega_i} = \sinh^2(gx) = G - 1, \quad (2.99)$$

describing how many idler photons appear at the output, relative to the number of signal photons incident on the input. It has fundamental consequences. Considering quantum fluctuations of half a photon on average incident on the input at both the signal and the idler frequency, the idler-to-signal conversion gain results in extra $(G - 1)/2 \approx G/2$ signal noise photons at the output. In this way, the signal-to-idler conversion gain allows a semi-classical interpretation of the quantum-limit on the added noise of an ideal parametric amplifier [5], further discussed in Sec. 4.6.

2.4.4 Phase-sensitive amplification in the degenerate case

A special case in parametric amplification occurs when the signal and the idler frequencies coincide, $f_s = f_i$, called the degenerate case [88]. In a 3WM parametric amplifier this occurs in the center of the gain band at $f_s = f_p/2$, far detuned from f_p .

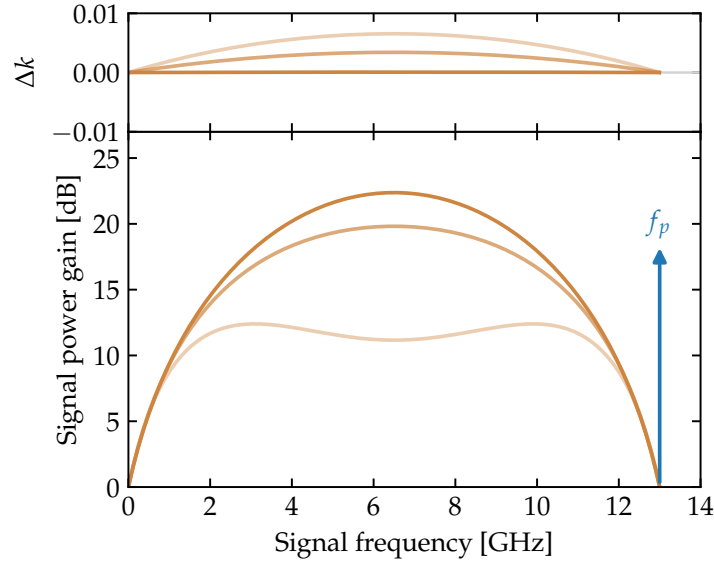


Fig. 2.18.: Frequency dependence of signal power gain with and without phase mismatch. Phase mismatch is positive for a convex dispersion relation. It reduces the achievable gain. The blue arrow marks the pump frequency. Parameters are $L_m = 80$ pH, $C_0 = 32$ fF, $C_J = 110$ fF and 220 fF for the curves with phase mismatch, and $C_J = 0$ for the curve without phase mismatch, $\beta = 0.125$, $\gamma = 0$, $N = 1000$, and $k_p A_{p0} = 0.8$ rad.

Since the signal and idler waves are identical, the boundary conditions for Eq. 2.90 change, having amplitude $A_s(x = 0) = A_{s0}e^{j\chi_{s0}}$, where χ_{s0} stands for the initial phase of the signal wave at $x = 0$, as above. Writing the phase of the pump wave explicitly for clarity, $A_p(x = 0) = A_{p0}e^{j\chi_{p0}}$, and following the same steps leading to Eq. 2.92, the complex signal amplitude is in the degenerate case

$$\begin{aligned} A_s(x) &= A_{s0} \left[e^{j(\chi_{s0})} \cosh(gx) + e^{j(\pi - \chi_{p0} - \chi_{s0})} \sinh(gx) \right] \\ &= A_s(0) \left[\cosh(gx) + e^{j\Delta\chi} \sinh(gx) \right], \end{aligned} \quad (2.100)$$

with the phase difference $\Delta\chi = \pi - \chi_{p0} - 2\chi_{s0}$. If this phase difference equals $\Delta\chi = 0$, the signal power gain is

$$G(\Delta\chi = 0) = [\cosh(gN) + \sinh(gN)]^2 \approx [2\cosh(gN)]^2 \quad (2.101)$$

where the latter approximation is valid for $G \gg 1$. This means that at $\Delta\chi = 0$ the gain is 6 dB larger than that of phase-insensitive parametric amplification. On the other hand, at $\Delta\chi = \pi$ the gain equals

$$G(\Delta\chi = \pi) = [\cosh(gN) - \sinh(gx)]^2 = e^{-2gN}, \quad (2.102)$$

exponentially decaying with circuit length. Setting again $\chi_{p0} = 0$, these two cases occur for $\chi_{s0} = \pi/2$ and $\chi_{s0} = 0$, respectively. In other words, one quadrature of the signal is amplified, while the other is deamplified. This has far-reaching consequences, such as the reduction of noise below the quantum limit of phase-insensitive amplification [5], called squeezing. It has been utilized experimentally both in resonant JPAs [44, 36] and in TWPAs [76, 89, 90]. Circuit simulations in this regime are discussed in Sec. 3.2.4.

2.4.5 Parametric amplification including phase mismatch

The case with nonzero phase mismatch Δk was treated analytically for an optical 3WM TWPA by Smith back in 1970 [91]. The modified solution for the signal and idler amplitudes is [57]

$$A_s(x) = A_{s0} e^{j\chi_{s0}} e^{j\Delta k x/2} \left[\cosh(gx) - j \frac{\Delta k}{2g} \sinh(gx) \right], \quad (2.103)$$

$$A_i(x) = 2A_{s0} e^{j(\pi - \chi_{s0})} e^{j\Delta k x/2} \frac{g_0 \omega_s}{g \omega_p} \sinh gx, \quad (2.104)$$

with the modified exponential gain factor

$$g = \sqrt{g_0^2(1 - \delta^2) - (\Delta k/2)^2}. \quad (2.105)$$

Then, the signal power gain is

$$\begin{aligned} G &= \cosh^2 gN + \frac{\Delta k^2}{4g^2} \sinh^2 gN \\ &= 1 + \sinh^2 gN \left(1 + \frac{(\Delta k/2)^2}{g^2} \right) \\ &= 1 + (g_0 N)^2 (1 - \delta^2) \frac{\sinh^2 gN}{(gN)^2}. \end{aligned} \quad (2.106)$$

In the limit of negligible phase mismatch, the gain simplifies to Eq. 2.94, increasing exponentially with circuit length N . Interesting is also the case $\Delta k/2 = g_0$, for which the gain increases quadratically with circuit length, $G \approx 1 + g_0^2 N^2 (1 - \delta^2)$, because $\sinh(gN)/gN \rightarrow 1$ [91]. A comparison of the signal power gain with and without phase mismatch is depicted in Fig. 2.18, along with the phase mismatch Δk .

2.4.6 Shortcomings of the coupled-mode theory

The coupled-mode theory is a powerful tool to achieve closed-form analytical expressions for a number of parametric processes, even including nonideal circumstances, such as phase mismatch. On the other hand, this theory is based on various assumptions and approximations, including the continuum approximation of the discrete circuit, the slowly-varying envelope approximation, the truncation of the Taylor series of the nonlinear current-phase relation, and several others. In a broadband device like a TWPA, however, the approximation with the strongest impact on the predictability of the results is the inclusion of only three modes in the theory (Sec. 2.4). Especially for TWPAs with low dispersion, a plethora of tones is generated, as depicted in Fig. 2.16, far beyond the three basic modes. In such cases, the three-mode CME model fails to predict the dynamics of the circuit.

This issue was analyzed by Dixon et al. [58]. They have shown that for the parameters suggested in Ref. [57] one has to include parametric processes up to fifth order (15 tones in total) to get results for parametric amplification close to those obtained by circuit simulations not making the above mentioned approximations. Not only is this approach tedious, it also negates the initial advantage of obtaining analytical expressions for the signal gain and other quantities, because these higher-order CMEs have to be solved numerically.

Moreover, there is another issue of more fundamental nature, originating in the generation of these additional, unwanted tones, particularly the up-conversion products with frequencies $f_p + f_s$ and $f_p + f_i$. Renberg Nilsson et al. have shown that in a homogeneous TWPA with low dispersion, $\omega \ll \omega_0, \omega_J$, the 3WM gain is limited by up-conversion, while for high dispersion, ω_p close to ω_c to inhibit up-conversion, the gain is limited by phase mismatch [86].

To unlock the full potential of 3WM TWPAs, dedicated techniques are required to solve both problems: Inhibiting unwanted parametric processes while simultaneously phase matching parametric amplification. These techniques are the focus of the next chapter. I will demonstrate that by adequate dispersion engineering techniques unwanted parametric processes can be suppressed to an extent that the dynamics of the basic three modes are described quantitatively by the three-mode CME model with analytical solutions.

Case studies on dispersion-engineered TWPAs via time-domain circuit simulations

Due to its broadband nature, a TWPA is a multimode device facilitating a multitude of parametric processes, beyond the three-mode model of Sec. 2.4. In general, it is therefore necessary to apply special techniques to achieve near-exponential amplification of the signal wave propagating along the circuit. An overview on these techniques, commonly referred to as dispersion-engineering techniques, is given in Sec. 3.2, together with a description of their differing purposes for 3WM and 4WM TWPAs. In this thesis, the open-source circuit simulator WRspice [92, 93] (Sec. 3.1) is utilized for a detailed analysis of TWPAs with various circuit architectures and dispersion-engineering techniques. Three examples are detailed in Sec. 3.2, followed by simulation studies on residual reflections (Sec. 3.3), saturation power (Sec. 3.4), and parameter spread in a TWPA (Sec. 3.5). These simulation studies form the basis of the experimental implementation of a dispersion-engineered TWPA presented in Chapter 4. The results presented in this chapter were published in part in Refs. [94] and [95].

3.1 WRspice circuit simulations of multimode parametric processes in TWPAs

Analytical or semi-analytical descriptions of TWPAs, such as the coupled-mode theory (Sec. 2.4), frequently employed in analyzing TWPAs [38, 87, 59, 57], offer valuable insights into the physics of such devices, but rely on several assumptions and approximations. These include transitioning from the discrete to the continuous telegrapher's equation [87], employing the slowly-varying amplitude approximation [59], and are usually limited to three forward-propagating waves,

neglecting reflections. The method proposed by Planat et al. [50], based on solving the discrete telegrapher's equation in the form of traveling waves, circumvents most of these approximations but does not accommodate including pump wave reflections (below it is demonstrated that pump reflections can be significant in TWPA circuits). Furthermore, it is a common approximation to neglect higher orders of nonlinearity in the theoretical description of the device under study (c.f. Eq. 2.11). Last but not least, these analytical theories can be extended to the multimode case, which is not only tedious [96, 58], but also negates the advantage of an analytical solution.

As an alternative, the circuit simulator WRspice [92, 93] (SPICE stands for Simulation Program with Integrated Circuit Emphasis) has been initially utilized by Naaman et al. to analyze impedance-engineered resonant JPAs [97] and by Dixon et al. to investigate multimode dynamics in a homogeneous TWPA without dispersion-engineering techniques [58]. In this thesis, WRspice is further employed to thoroughly analyze TWPAs with various circuit architectures and dispersion-engineering techniques, allowing for the extraction of numerous metrics and quantities [94, 95]. This approach stands out by its flexibility in modeling complicated circuits like dispersion-engineered TWPAs, including nonideal circuit elements (Sections 2.1.3 and 4.2.1), or real-world effects such as parameter spread (Sec. 3.5) [95]. Furthermore, WRspice takes into account the full current-phase relation of Josephson junctions. The accuracy in predicting device behaviour, even in the presence of reflections and multimode interactions, is demonstrated by the remarkable agreement between WRspice simulations and experiment, as shown in Figures 4.15 and 4.19 in Chapter 4.

All circuit simulations presented in this thesis utilize time-domain simulations with the WRspice transient solver. A frequency-domain alternative is the harmonic-balance method, recently applied to parametric amplifiers by several groups [98, 99, 100, 101, 86]. Both methods are compared at the end of this Section.

3.1.1 Simulation setup

WRspice is a circuit simulation software, based on Berkeley SPICE3, capable of simulating circuits containing Josephson junctions [92, 93]. In this work, the default Josephson junction model, based on the SPICE implementation of the RCSJ model by Jewett [102] is used, denoted as `jj(level=1)` in a netlist (Fig. 3.1). Further, the default piecewise-linear quasiparticle resistance model, `rtype=1`, is used. Other more advanced models provided by WRspice differ mainly in the modeling of the nonlinear quasiparticle resistance, but showed no difference in rf-SQUID-based circuits, where the Josephson junction remains in the zero-voltage state. The

three main parameters i_{crit} , v_m , cap denote the critical current I_c , the product of critical current and subgap resistance $I_c R_{sg}$, and the junction capacitance C_J , respectively.

Additional to currents and voltages, WRspice calculates superconducting phases. A transient analysis starts from a default initial condition of setting the values of all voltage and current sources to zero, thereby satisfying the condition that the sum of phase differences around loops of inductors and Josephson junctions remains zero (which is a multiple of 2π). From this initial condition, sources are ramped up to their target values, and phases and fluxes evolve according to the dynamics of the circuit.

Two-port simulation framework

Throughout this thesis, simulations were performed with the schematic presented in Fig. 3.1a. This circuit enables time-domain simulations of a two-port, excited by a pump current $i_{p,q}(t) = I_{p,q} \sin(\omega_p t)$ and a signal current $i_{s,q}(t) = I_{s,q} \sin(\omega_s t)$. The index q denotes the respective source current, injecting a pump current $i_p = i_{p,q}/2$ and a signal current $i_s = i_{s,q}/2$ to the two-port in the case of impedance-matching to the source and load impedance $Z_0 = 50 \Omega$. Optionally, an initial phase difference between the two sinewaves can be specified, e.g. to investigate phase-sensitive parametric amplification (Sec. 3.2.4). To simulate amplification of backward-propagating waves (Sec. 3.3.2), or forward-backward-coupled conversion gains (Sec. 4.6.6), the signal source can be placed at the nominal output of the two-port. Flux-biasing of the rf-SQUIDs is achieved via current-biasing with a constant current I_{dc} supplied via a piecewise-linear current source (PWL). To avoid large transients, this current is ramped up from zero within 0.4 ns.

A typical schematic of an rf-SQUID-based TWPA circuit and its corresponding netlist are presented in Fig. 3.1b. All unit cells of the TWPA ($N = 1500$ in the example) are modeled individually to allow modification of their circuit parameters, for example to simulate the effect of parameter spread on the dynamics of the circuit (Sec. 3.5). Inserting voltage sources with a value of zero volts allows extracting the current flowing through the source without modifying the behavior of the circuit. Currents and voltages of all nodes connecting adjacent TWPA unit cells are extracted and saved to files for post-processing. Instead of the ideal rf-SQUIDs modeled in Fig. 3.1b, more complex circuits, consisting of nonideal rf-SQUIDs with parasitic circuit elements, and containing transmission lines, can be modeled in a similar fashion (Sec. 4.2.5). Moreover, using the framework in Fig. 3.1a, a one-port circuit element like a nonideal rf-SQUID can be simulated by configuring it to a two-port as shown in Fig. 3.1c. This is helpful, for example, to investigate the

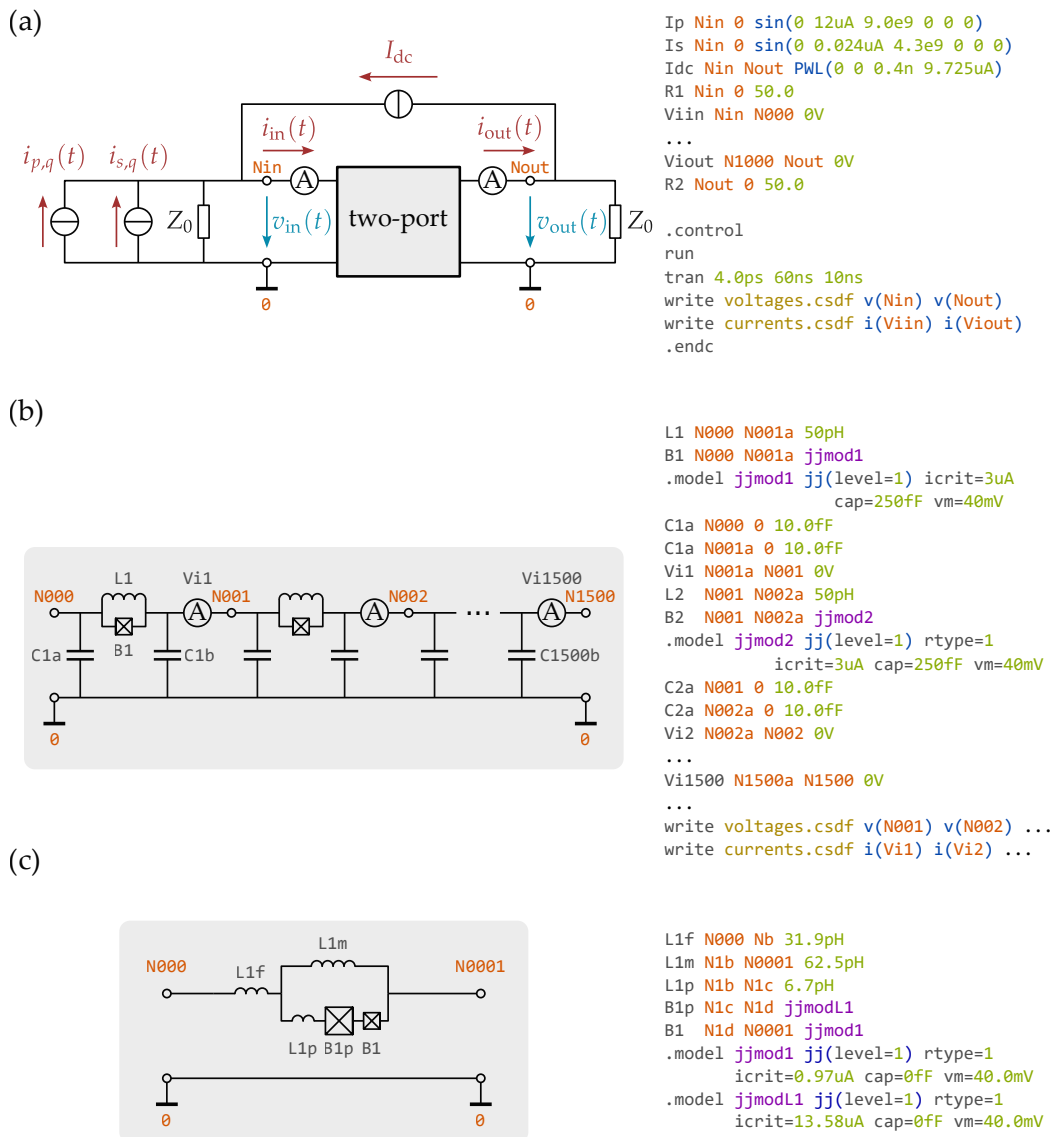


Fig. 3.1.: Schematics and netlist snippets of time-domain TWPA simulations in WSpice. **(a)** Circuit implementation and boundary conditions of nonlinear two-port simulations and the corresponding netlist snippet with representative currents and frequencies. Pump, signal, and bias currents are injected into the two-port. The voltages and currents at the two ports are extracted and saved to files for post-processing. Reference impedance is $Z_0 = 50 \Omega$. **(b)** Schematic representing a TWPA with $N = 1500$ unit cells. Each unit cell contains a zero-voltage source to extract the current. Node voltages at indicated nodes are extracted, too. **(c)** Instead of a TWPA, other two-ports can be simulated in the same way, such as a single nonideal rf-SQUID, e.g., to determine its large-signal inductance.

inductance of a SQUID for large signal amplitudes, which presents a rather tedious task via analytic means. Such simulations were performed for the input-power calibration discussed in Sec. 4.3.

Transient solver control line

The appropriate setup of the transient analysis, parametrized by the `tran` line, depends on both the circuit properties and the envisaged post-processing. Representative time traces of currents through three nodes in the TWPA circuit discussed in Sec. 3.2.4 are presented in Fig. 3.2a. The beginning of the time traces is dominated by the transient response of the circuit to the ramped-up bias current. Ignoring dispersion, propagating wavefronts reach the opposite end of the circuit with a time delay of approximately $t_{\text{delay}} = \ell/v_{\text{ph}} \approx N\sqrt{LC_0}$, where L denotes the SQUID inductance, and $C_0 = \bar{C}$ is the average ground capacitance of the circuit. Since the initial flux in all rf-SQUIDs is zero, the SQUID inductances, $L = L_m/(1 + \beta_L)$, are smaller than at their nominal biasing around the Kerr-free point, where $L = L_m$. After the first bias-current wavefront the current equals $I_{\text{dc}}/2$, also causing $L < L_m$. Hence, the first two wavefronts propagate faster than all subsequent. After the bias current settled to its steady state, I_{dc} , the time delay is $t_{\text{delay}} \sim 3$ ns.

Regarding the pump wave, most of the circuit dynamics stabilize after the first reflection of the pump wave from the nominal output has reached each node in the circuit. This is particularly pronounced for the TWPA circuits with periodic capacitance variation, discussed in Sec. 3.2, where circuit dynamics rely on multiple reflections. In Fig. 3.2a this can be observed at ca. 5 ns at node $n = 800$ and 7 ns at $n = 1400$. After 10 ns circuit dynamics are close to their steady state. After this settling time of 10 ns, time-trace data is saved to files for further processing, omitting data of the transients before 10 ns. Although the circuit is not yet fully at its steady state at 10 ns, choosing a larger settling time, e.g. 20 ns, did not result in notably different results justifying the higher computational cost. For longer circuits, e.g. for the simulations of TWPAs with $N = 1900$ or 2352 unit cells in Chapter 4, a longer settling time of 20 ns is chosen.

The total sampling time and the time step determine the properties of the Discrete Fourier Transform (DFT) in post-processing. In most simulations presented in this thesis, data is saved between 10 ns and 60 ns to achieve a total sampling time of $t_{\text{sample}} = 50$ ns, which determines the resolution of the DFT, $\Delta f = 1/t_{\text{sample}} = 20$ MHz. The maximum time step is set to $\Delta t = 4$ ps, small enough to override the adaptive time-step control of WRspice, resulting in equidistant time series. The maximum frequency of the DFT is set by the time step to $f_{\text{max}} = 1/\Delta t = 125$ GHz. This value is larger or approximately equal to the cutoff frequency of

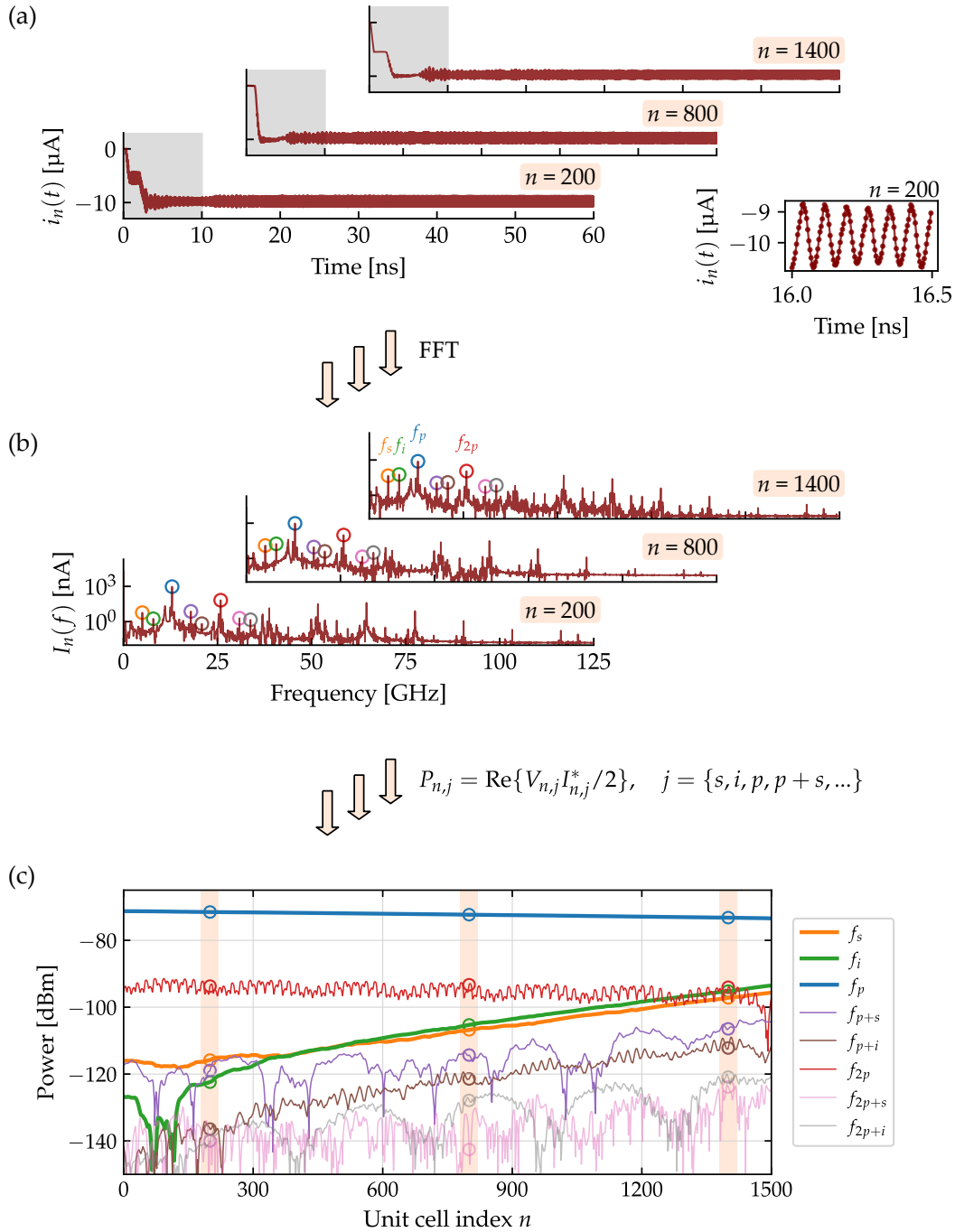


Fig. 3.2.: Workflow for extracting multimode dynamics from time-domain simulations. **(a)** Representative time series of node currents (see Fig. 3.1b) from a TWPA simulation. Node voltages are also extracted but not shown. The first ~10 ns exhibit strong transients and are discarded for post-processing. The inset shows a close-up view of the discrete time series. **(b)** In post-processing, the time series are Fourier-transformed to the frequency domain. Voltages and currents at relevant frequencies are extracted (colored circles) to analyze the multimode dynamics inside the TWPA. **(c)** Evolution of wave powers along all $N = 1500$ unit cells of the circuit. Nodes with time series and spectra shown in (a) and (b) are highlighted. Data from mPCV-TWPA simulations (Sec. 3.2.4), at $f_s = 5$ GHz, $f_p = 12.92$ GHz, $I_p = 1.8$ μA, $I_s = 0.01$ μA, and $I_{dc} = -9.8$ μA.

most investigated TWPA circuits. Additionally, it is sufficiently large to satisfy the Shannon-Nyquist sampling theorem for up to the fourth pump harmonic.

3.1.2 Post-processing

Figure 3.2 illustrates the workflow of post-processing, with the aim of extracting complex voltages and currents for all relevant tones from the time-domain data of transient simulations. First, the time series of node voltages $v_n(t)$ and currents $i_n(t)$ for each node $n = 0 \dots N$ are transformed to the frequency domain (Fig. 3.2b). This transformation is performed using the Fast Fourier Transform (FFT) as an implementation of the DFT,

$$I_{n,q} = \frac{1}{M} \sum_{\tau=0}^{M-1} i_n \exp \left(-\frac{2\pi j}{M} \tau q \right), \quad q = 0, \dots, M-1, \quad (3.1)$$

$$V_{n,q} = \frac{1}{M} \sum_{\tau=0}^{M-1} v_n \exp \left(-\frac{2\pi j}{M} \tau q \right), \quad q = 0, \dots, M-1, \quad (3.2)$$

where M denotes the size of both the time and frequency series, and the index q defines the corresponding frequency bins, $f_q = q\Delta f$. In setting up the simulation, t_{sample} is chosen such that relevant frequencies f_j with index $j \in \{s, i, p, p+s, p+i, 2p, \dots\}$ coincide with discrete frequency bins after the DFT. For example, when $t_{\text{sample}} = 50$ ns, frequency bins are spaced by $\Delta f = 20$ MHz, and signal and pump frequencies are chosen as multiples of Δf , so that their mixing products are also multiples of Δf . From the discrete sets of complex currents and voltages, the values at indices p corresponding to all relevant tones are extracted for each node n , $I_{n,j} = I_{n,q}(f_q = f_j)$ and $V_{n,j} = V_{n,q}(f_q = f_j)$, illustrated as colored circles in Fig 3.2b. Note the presence of pump harmonics and up-conversion products of signal and idler, up to high orders.

Apart from these expected tones, the spectrum contains a high baseline with approximately Lorentzian shape centered around pump harmonics, and other additional peaks, both limiting the dynamic range of Fourier-transformed transient simulations. While the underlying mechanisms are not yet fully understood, the peaks centered around the pump and its harmonics may be attributed to interactions of the pump and the stopband (Sec. 3.2.4), and the high baseline with Lorentzian shape (c.f. Ref. [103]) to deviations from the steady state, since it corresponds to an exponential decay in the time domain. The latter could be mitigated by allowing longer settling times, or by applying appropriate window functions, but it does not present a severe limitation for the simulations presented in this work.

Afterwards, various quantities of interest can be calculated using the complex currents and voltages, as presented next. As an example, Fig 3.2c shows the evolution of the powers $P_{n,j} = \text{Re}\{V_{n,j}I_{n,j}^*/2\}$ of various waves propagating along the TWPA, visualizing multimode dynamics (further explained in Sec. 3.2).

3.1.3 Extracted quantities

Quantities extracted from transient circuit simulations, as presented above, include but are not limited to powers and power gains, impedances, inductances, wavenumbers, scattering (S) parameters (and other two-port parameters), power-wave quantities, and conversion gains (or X-parameters, as the nonlinear extension to S-Parameters [104, 105]). For readability the frequency index j for frequency f_j is omitted, if not stated otherwise.

Power

The complex power at node n is defined as

$$S_n = P_n + jQ_n = \frac{1}{2}V_n I_n^*, \quad (3.3)$$

where P is the active and Q the reactive power, and the coefficient $1/2$ takes into account that complex voltages and currents are defined as peak values here. Particularly important is the active power, stating the power carried by the wave, and is positive for forward-propagating waves (from nominal input to the nominal output). However, in the presence of impedance mismatches and reflections, power-wave quantities are better suited for describing forward and backward propagating (or transmitted and reflected) waves.

Power-wave quantities and scattering parameters

In the two-port simulation framework as defined in Fig. 3.1a, the output current has an opposed sign compared to common definitions in the literature [85]. To comply with common definitions, we define $I_2 = -I_{\text{out}}$, and the other quantities as $I_1 = I_{\text{in}}$, $V_1 = V_{\text{in}}$, and $V_2 = V_{\text{out}}$. Then, the incident and outgoing power-wave quantities at port $q \in \{1, 2\}$ are defined as

$$a_q = \frac{V_q + Z_0 I_q}{2\sqrt{Z_0}}, \quad (3.4)$$

$$b_q = \frac{V_q - Z_0 I_q}{2\sqrt{Z_0}}, \quad (3.5)$$

with reference impedance Z_0 . The corresponding scattering matrix

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad (3.6)$$

contains the scattering parameters (S-parameters), with the input reflection coefficient and forward transmission coefficient given by

$$S_{11} = \frac{b_1}{a_1} = \frac{V_1 - Z_0 I_1}{V_1 + Z_0 I_1} = \frac{V_{in} - Z_0 I_{in}}{V_{in} + Z_0 I_{in}}, \quad (3.7)$$

$$S_{21} = \frac{b_2}{a_1} = \frac{V_2 - Z_0 I_2}{V_1 + Z_0 I_1} = \frac{2V_{out}}{V_{in} + Z_0 I_{in}}, \quad (3.8)$$

in the case of excitation at the input port 1 (Fig. 3.1a), and $a_2 = 0$ because of $V_{out} = Z_0 I_{out}$.

Furthermore, relating power-wave quantities at different frequencies to each other, conversion gains (or intermodulation gains) can be calculated, as defined in Sec. 4.6.3.

Signal power gain and transducer gain

All WRspice simulations presented in this thesis use a source (Z_S) and load (Z_L) impedance equal to the reference impedance $Z_S = Z_L = Z_0$. However, the characteristic impedance Z of the TWPA is generally not equal to Z_0 and may, furthermore, be not purely real-valued but possess a non-negligible imaginary part. The slight impedance mismatch, with $Z \neq Z_0$, introduces ambiguity regarding the definition of the amplifier's two-port power gain. The power gain defined as $G_P = P_{out}/P_{in}$, as often done in literature on TWPA modeling, is valid only in the impedance-matched case, $Z = Z_0$, or $S_{11} = 0$. Note that the latter condition is usually not fulfilled in many practical TWPAs, where even $S_{11} > 0$ dB is possible, see Sec. 3.3. Ignoring this, the gain can be severely overestimated, because this definition does not differentiate between forward and backward propagating waves. It can be shown that $G_P = |S_{21}|^2 / (1 - |S_{11}|^2)$ for $Z_S = Z_L = Z_0$ [85].

Taking into consideration reflections of the signal wave at the input and output, the definition of the transducer power gain [85] is employed. This is the most general power gain definition in a two-port amplifier, representing actual performance in a system, as experimentally measured with a vector network analyzer. It relates the power dissipated in the load, P_L , to the power available from the source, P_A , and is

$$G = P_L / P_A = |S_{21}|^2, \quad (3.9)$$

under the condition of $Z_S = Z_L = Z_0$.

Wavenumber

The average dimensionless wavenumber k can be calculated from S_{21} as the average phase shift per unit cell of a transmitted signal,

$$k = -\frac{\arg(S_{21})}{N}. \quad (3.10)$$

Large-signal inductance of a one-port inductive element

Finally, the impedance of a one-port, such as the SQUID shown in Fig. 3.1, can be found as

$$Z_{SQ,j} = R_{SQ} + j2\pi f_j L_{SQ} = (V_{in,j} - V_{out,j}) / I_{in,j}, \quad (3.11)$$

yielding the SQUID inductance at frequency f_j depending on the current amplitude and the bias current I_{dc} .

3.1.4 Shortcomings of time-domain simulations in WRspice and comparison to other simulation packages

Transient circuit simulations provide a powerful and extremely flexible tool for analyzing complex and broadband circuits such as TWPAs. This comes at the price of computational cost, since each data point requires a dedicated transient simulation. For example, the simulation presented in Fig. 3.2 took 15 minutes running on a single CPU of a desktop PC with a Intel® Core™ i5-6500 and 8 GB of RAM available, or 7 minutes on average when performing 32 simulations of the same circuit running in parallel on 32 CPUs on a workstation with an AMD Ryzen™ Threadripper™ 3990X with 128 GB of RAM. In both cases, WRspice was running in a Windows Subsystem for Linux (WSL) in Microsoft Windows® (it is recommended to run WRspice in a Linux® environment), controlled by a parallelized python script.

Apart from computational cost, the WRspice based approach presented here has other shortcomings. The main drawback of time-domain simulations in WRspice is the absence of frequency-dependent resistors, required to model, for example, frequency-dependent port impedances, emulating exterior circuitry such as isolators, or frequency-dependent dielectric loss. The latter is usually modeled via a frequency-dependent resistor $R_{C,p} = 1/(\omega C \tan \delta)$ in parallel to a capacitor C with loss tangent $\tan \delta$ (see Sec. 4.2.5). In other SPICE-based simulators, this is possible via applying Laplace transform [106], which is currently not supported in WRspice.

Alternatively, frequency-dependent resistors can be emulated for the use in transient circuit simulators via cascaded RC circuits consisting of constant resistors and capacitors [107]. Furthermore, concerning narrowband resonant devices, such as JPAs or dimer Josephson junction array amplifiers (DJJAAs) [66], transient circuit simulations may be relatively inefficient, since a long settling time has to be awaited due to high quality factors of the resonances.

An alternative to Fourier-transformed time-domain simulations are techniques operating directly in the frequency-domain. Since the standard AC solver of SPICE simulators performs linearization around a DC operating point, it cannot be used to simulate parametric effects (except using the pumpistor model [108, 101]). Simulations of nonlinear frequency-converting processes are possible via the harmonic balance method, which allows including a predefined number of harmonics and mixing products to the analysis. It has recently been used by several groups in the context of superconducting parametric amplifiers [98, 99, 100, 101, 86]. Noteworthy is the open-source package `JosephsonCircuits.jl`, particularly designed for simulating scattering parameters and quantum efficiency of JPAs and JTWPAs [109, 98, 99]. While Peng et al. state that their simulator package `JosephsonCircuits.jl` is 10,000 times faster than `WRspice` for their TWPA circuits with 2047 and 3998 Josephson junctions [99], Levochkina et al. report a speed-up of about five times for another benchmark TWPA circuit with 250 SNAILs [103], indicating different scaling with circuit complexity. Although `JosephsonCircuits.jl` is much faster than `WRspice` in simulating 4WM TWPAs, it does not support flux-biasing and 3WM at the current stage. Further quantitative comparison of `WRspice` to other simulators, including transient and harmonic balance simulators, in the context of TWPAs is provided in Refs. [101, 99, 103].

3.2 Dispersion engineering

3.2.1 Overview on dispersion engineering techniques

To unlock the full potential of TWPAs for high and broadband signal gain, exponentially scaling with circuit length, special techniques are necessary, in the following referred to as dispersion engineering techniques. In a simplified picture, the main purposes of dispersion engineering can be summarized in the following way:

- In 4WM, all parametric processes are inherently phase-mismatched. The purpose of dispersion engineering is the phase matching of the intended parametric process.
- In 3WM, many parametric processes can be inherently phase-matched. The purpose of dispersion engineering is the mismatching of all unwanted processes while keeping the intended process phase-matched.

The inherent phase-mismatch in 4WM TWPAs stems from the Kerr effects of self-phase modulation (SPM) and cross-phase modulation (XPM), causing a nonlinear phase mismatch $\Delta k_{\text{NL}} = 2\vartheta_p + \vartheta_s + \vartheta_i$ (Eqs. 2.82–2.85), such that $\Delta k = 2k_p - k_s - k_i + \Delta k_{\text{NL}}$ for basic 4WM with the frequency relation $2f_p = f_s + f_i$. Even in the case of negligibly small linear phase mismatch $\Delta k_{\text{L}} = 2k_p - k_s - k_i$, 4WM parametric amplification is strongly mismatched by Δk_{NL} . The same happens for other parametric processes such as frequency conversion, which are thus inherently suppressed. Therefore, to achieve gain exponentially scaling with device length, phase matching must be achieved by compensating the total phase mismatch Δk [56, 59].

On the other hand, in 3WM, the Kerr effects SPM and XPM are small, or, in the Kerr-free case, vanish [57]. For small frequencies $\omega \ll \omega_0, \omega_J$ the dispersion of a homogeneous transmission line is small, $k \approx \omega / \omega_0 \propto \omega$ (c.f. Eq. 2.47). Therefore, for the basic 3WM parametric amplification process, $f_p = f_s + f_i$, the phase-matching condition $k_p = k_s + k_i$ is approximately fulfilled. At the same time, however, the phase mismatches of unintended 3WM processes, for example

$$f_{p+s} = f_p + f_s, \quad \Delta k_{p+s} = k_{p+s} - k_p - k_s, \quad (3.12)$$

$$f_{p+i} = f_p + f_i, \quad \Delta k_{p+i} = k_{p+i} - k_p - k_i, \quad (3.13)$$

$$f_{2p} = 2f_p, \quad \Delta k_{2p} = k_{2p} - 2k_p, \quad (3.14)$$

are also small, leading to a multitude of parametric processes occurring simultaneously in a 3WM TWPAs [58]. Such multimode dynamics can be seen, for example, in Fig. 3.5. These modes are unwanted in the context of parametric amplification,

because they consume power of the pump, signal, and idler waves, ultimately limiting the achievable signal gain. Therefore, to achieve high 3WM gain, such unintended parametric processes must be suppressed, usually by increasing their phase-mismatches, while retaining phase matching of the basic 3WM process. In the case of parametric amplification, all unwanted processes involve frequencies higher than the pump frequency. Therefore, their suppression is possible by engineering sufficiently large dispersion at frequencies above the pump frequency. Both phase matching and the suppression of unwanted modes are further explained in Sec. 3.2.2 on a concrete example.

Various techniques for phase matching or phase mismatching in TWPAs are summarized in Tab. 3.1. The most popular techniques are resonant phase matching (RPM) and periodic circuit parameter variation, implemented for example as periodic capacitance variation (PCV), where the ground capacitances of the transmission line are varied periodically. Both methods create a narrowband dispersive feature in the dispersion relation $k(\omega)$, which is used to shift the phase of the wavenumber k_p of the pump wave by an appropriate choice of f_p to fulfill the respective phase-matching condition.

In the following sections, three dispersion-engineering implementations for an rf-SQUID-based 3WM TWPAs are presented and their effect on multimode dynamics is illustrated. The first and the second involve lowering the plasma frequency to induce strong dispersion above the pump frequency and suppress unwanted parametric processes. Phase matching for the intended parametric amplification is achieved by RPM and PCV, respectively. The third approach introduces a variant of PCV, which we call multiperiodic capacitance variation (mPCV). In this method, the ground capacitors are varied in two periods. One period creates a wide stopband inducing strong dispersion above the pump frequency to suppress unwanted processes, while the other period creates a narrow stopband, slightly below the pump frequency, retaining phase matching for parametric amplification.

3.2.2 Resonant phase matching

Circuit description and resulting gain profile

The basic circuit architecture is a transmission line using nonhysteretic rf-SQUIDs as nonlinear inductive elements (see Sec. 2.1.2) arranged with capacitors to ground (Fig. 3.3). A magnetic field $\vec{B}$, or alternatively, a DC current injected into the circuit, is used to flux-bias the rf-SQUIDs such that the Kerr nonlinearity coefficient γ is zero, and the coefficient of quadratic nonlinearity, β , is close to its maximum. The small-signal inductance is equal to L_m at this flux bias, and the ground capacitors C_0 are chosen such that the characteristic impedance of the transmission line is $Z =$

$\sqrt{L_m C_0} = 50 \Omega$, impedance-matched to the source and load impedance, terminating the TWPA circuit.

The junction capacitances C_J lead to dispersion $k(\omega)$ according to Eq. 2.47, as illustrated in Fig. 3.4a. In the RPM variant of the TWPA, resonators are capacitively coupled to the transmission line at regular intervals of m unit cells. These resonators can be implemented either as lumped-element resonators, with capacitance C_r and inductance L_r , or as distributed resonators, e.g. transmission-line resonators, with a length equal to a quarter wavelength at resonance. Including the resonators to the circuit results in a nonmonotonic dispersive feature in $k(\omega)$ around the resonance frequency, as shown in the inset of Fig. 3.4a.

The impact of incorporating RPM elements into the TWPA circuit is depicted by the solid and dash-dotted gray lines in Fig. 3.4b, representing the TWPA with and without RPM, and otherwise identical parameters. The graphs are obtained by circuit simulations as described in Sec. 3.1 with the circuit parameters listed in Tab. 3.2, but $C_J = 40$ fF. Inserting resonators to the circuit allows lowering k_p

Tab. 3.1.: Overview on dispersion-engineering techniques. Purposes are phase matching of parametric amplification (PA), and suppression of pump harmonics and unwanted mixing products.

Name, Refs.	Purpose
Periodic loading [38, 96], Periodic circuit parameter variation [110, 50, 76, 77], this work [95]	Phase matching of PA by increasing k_p (4WM) / lowering k_p (3WM), suppression of pump harmonics coinciding with stopbands
Multiperiodic capacitance variation, this work [94]	Phase matching of PA by increasing k_p (4WM) / lowering k_p (3WM), suppression of pump harmonics and higher-frequency mixing products by increased dispersion
Resonant phase matching [59, 49, 111, 110, 90], this work [95]	Phase matching of PA by increasing k_p (4WM) / lowering k_p (3WM)
Plasma-frequency engineering [60, 112], this work [95]	Suppression of higher-frequency modes by increased dispersion
Cutoff-frequency engineering [111, 86]	Suppression of higher-frequency modes by preventing their propagation
Quasi-phase matching [60]	Phase matching of PA via periodical signal-phase jumps by π due to sign reversal of quadratic-nonlinearity coefficient (3WM only)
Reversed-Kerr phase matching [53]	Phase matching of PA by compensating the linear with the nonlinear phase mismatch due to reversed sign of Kerr coefficient (4WM only)
Self-phase matching in left-handed transmission line [113]	Phase matching of PA by compensating the linear with the nonlinear phase mismatch due to reversed sign of group velocity (4WM only)

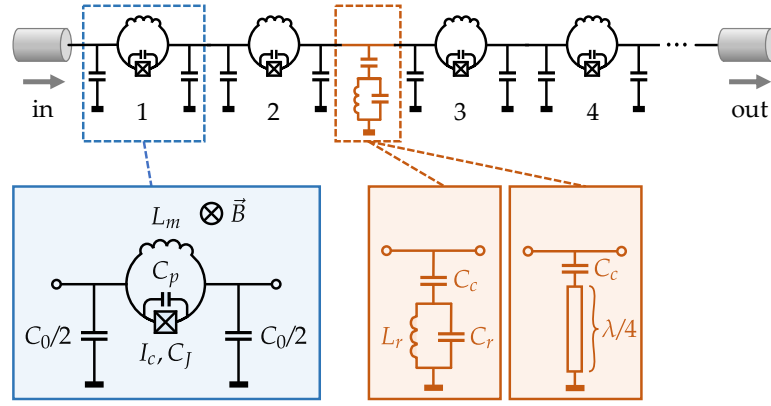


Fig. 3.3.: Scheme of an rf-SQUID-based TWPA with resonant phase matching (RPM) and low plasma frequency. Shown is one section of $m = 4$ identical unit cells (blue-shaded panel). The parallel capacitor C_p symbolizes an enlarged junction capacitance C_J to reduce the plasma frequency. Each section includes a resonator for phase matching, either lumped-element or distributed (brown-shaded panels), capacitively coupled to the transmission line. Ground capacitors adjacent to the resonator are decreased to $(C_0 - C_c)/2$.

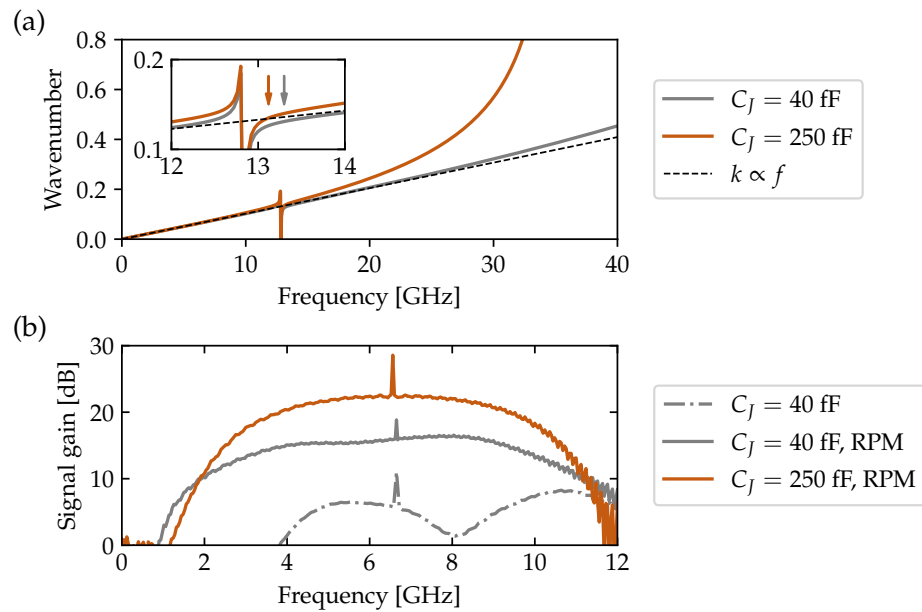


Fig. 3.4.: Wavenumber and signal gain of the RPM-TWPA (Fig. 3.3). **(a)** Increased dispersion by enlarged C_J to suppress unwanted parametric processes, utilizing the nonmonotonic resonant feature for phase-mismatch compensation of parametric amplification (inset). **(b)** WRspice simulations of the resulting signal gain in three scenarios: no dispersion engineering (gray dash-dotted), RPM (gray solid), and RPM with increased C_J (brown). The peaks at the centers of all gain profiles arise from phase-sensitive amplification at half the pump frequency ($f_s = f_p/2$). Circuit parameters see Table 3.2. Simulations use a pump current of $6.59 \mu\text{A}$, pump frequencies 13.3, 13.3, 13.12 GHz in the order of the legend (arrows of same colors in (a)), bias current of $7.5 \mu\text{A}$, and signal current of $0.01 \mu\text{A}$.

Tab. 3.2.: Circuit parameters used for transient circuit simulations

	RPM-TWPA	PCV-TWPA	mPCV-TWPA
Inductance	$L_m = 80$ pH	$L_m = 80$ pH	$L_m = 84$ pH
Critical current	$I_c = 1.03$ μ A	$I_c = 1.03$ μ A	$I_c = 1.57$ μ A
Junction capacitance	$C_J = 250$ fF	$C_J = 100$ fF	$C_J = 20$ fF
$I_c R_J$ product	$I_c R_J = 40$ mV	$I_c R_J = 40$ mV	$I_c R_J = 16.5$ mV
Screening parameter	$\beta_L = 0.25$	$\beta_L = 0.25$	$\beta_L = 0.76$
Resonator capacitance	$C_r = 1$ pF		
Resonator inductance	$L_r = 150$ pH		
Coupling capacitance	$C_c = 20$ fF		
Unit cells per resonator	$m = 4$		
Ground capacitance	$C_0 = 32$ fF	$C_0 = 32$ fF	$C_0 = 40$ fF
Variation depth		$\zeta = 0.3$	$\zeta_1 = 0.12, \zeta_2 = 0.78$
Unit cells per period		$m = 24$	$m = 20$
Number of unit cells	$N = 1000$	$N = 1488$	$N = 1500$

to compensate the phase mismatch due to the slight dispersion of $k(\omega)$, which increases the signal gain.

Further enhancement of the signal gain focuses on mitigating unwanted parametric processes. To accomplish this, the effective SQUID capacitance C_J is increased. This can be done for example by introducing a capacitor C_p in parallel with the Josephson junction, or by lowering the critical current density of the Josephson junctions in the fabrication process. Increasing C_J increases the dispersion (brown line in Fig. 3.4a) and induces a phase mismatch for mixing processes involving frequencies higher than f_p [60]. The resulting phase mismatch Δk for the basic parametric amplification process can then be compensated by lowering k_p at the RPM feature by fine-tuning f_p (brown arrow). The effectiveness of this strategy is demonstrated by the solid brown line in Fig. 3.4b, indicating more than 20 dB gain within a 3-dB bandwidth of more than 5 GHz. Increasing C_J beyond 250 fF does not result in significantly higher gain but leads to stronger gain ripple due to alterations in the line impedance (Eq. 2.44). The distinct peaks in the centers of all three gain profiles (at $f_s = f_p/2$) are discussed in Sec. 3.2.4.

Multimode dynamics for weak dispersion

Figure 3.5 showcases the multimode processes that result in the three signal-gain profiles shown in Fig. 3.4b. It depicts the spatial evolutions of the powers of pump, signal, and idler waves, along with parasitic waves, such as the second harmonic of the pump and the first six sideband modes, for a signal frequency of 6.8 GHz.

When the dispersion is weak (Fig. 3.5a), many 3WM processes are approximately phase-matched. For example, the phase mismatch of second-harmonic generation (Eq. 3.14) corresponds to a coherence length $\xi_{2p} = \pi/\Delta k_{2p}$ of ca. 200 unit cells. On this length, the second harmonic can grow to amplitudes exceeding that of the pump. After this coherence length, the direction of power transfer reverses, and power is transferred back to the pump. However, harmonic generation occurs as a cascaded process, generating numerous harmonics, which all have different coherence lengths. Therefore, not the whole power is transferred back to the pump, which reduces in amplitude the farther it propagates. As a result, the pump, required to drive parametric amplification, is substantially reduced in its average amplitude. Similar reasons enable significant power transfer between the signal and idler waves and their parasitic sidebands, particularly those of first order, f_{p+s} and f_{p+i} . These sidebands grow to amplitudes larger than those of signal and idler, ultimately limiting signal growth.

Additionally, parametric amplification is affected by a small phase mismatch Δk , corresponding to a coherence length $\xi = \pi/\Delta k$ on the order of a few hundred unit cells. Since $\xi < N$, the phase mismatch reduces the achievable signal gain. This obstacle is solved by inserting RPM to the circuit, resulting in the multimode dynamics shown in Fig. 3.5b. By decreasing k_p , the phase mismatch is approximately compensated, such that $\xi \gg N$. Simultaneously, by decreasing k_p , the phase mismatch of Eq. 3.14 is increased ($\xi_{2p} \approx 150$ unit cells), and the second harmonic of the pump is slightly suppressed compared to the case without RPM. This, together with better phase matching, leads to higher signal gain when RPM is used.

Close-to-ideal parametric amplification via dispersion engineering

Almost pure 3WM can be achieved by strong dispersion, as shown in Fig. 3.5c. In this case, the dispersion is so strong, that the coherence length of second-harmonic generation reduces to ca. 25 unit cells, effectively limiting its growth to values more than 10 dB below the pump amplitude. The third harmonic (not shown) is more than 30 dB lower than the pump. Consequently, the power transfer of the pump to its harmonics is almost negligible, and the pump amplitude remains constant along the TWPA, leaving dissipation aside. Since the first-order sidebands, f_{p+s} and f_{p+i} , occur at lower frequencies than f_{2p} , they experience less dispersion, and the phase mismatches of their generating processes are lower. Nevertheless, dispersion is sufficient to suppress the first-order sidebands to amplitudes ca. 5 dB below the signal and idler, so their parasitic effect is much reduced. Negligible is the effect of higher-order sidebands, in contrast to the case with weak dispersion, where they were of similar amplitude as signal and idler. The growth of the signal wave is close

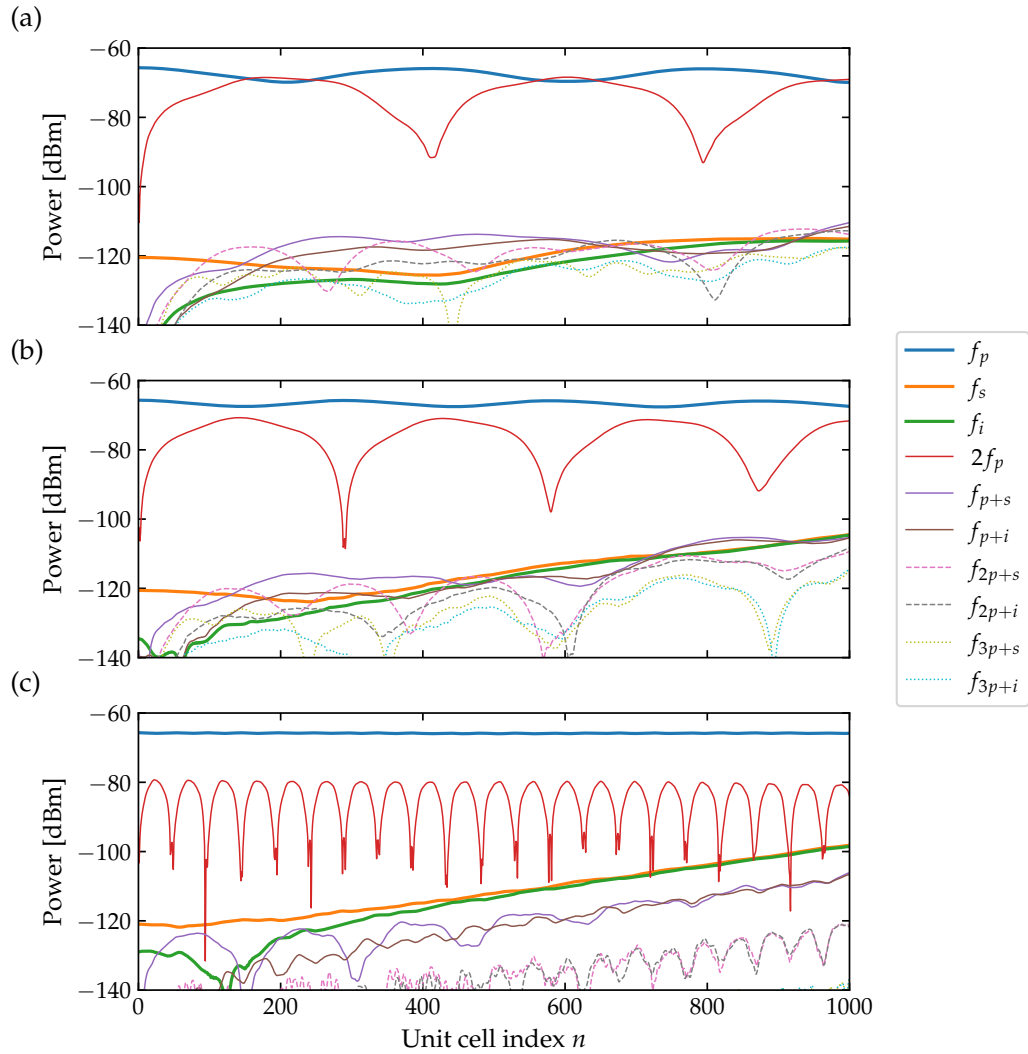


Fig. 3.5.: Effect of dispersion engineering on the spatial evolution of mode powers. Shown are the three scenarios of Fig. 3.4 at $f_s = 6.8$ GHz. Modes propagate from left to right. **(a)** $C_J = 40$ fF (weak dispersion), no dispersion engineering. Power is transferred alternately between the pump (blue) and its second harmonic (red), leading to strong reduction of the average pump amplitude. Power transfer from signal (orange) and idler (green) to various sidebands limits the signal gain to ca. 5 dB. **(b)** $C_J = 40$ fF, RPM. Lowering k_p by RPM leads to larger phase mismatch in second-harmonic generation, hindering its growth. Higher signal gain is achieved. **(c)** $C_J = 250$ fF, RPM. Strong dispersion at higher frequencies leads to large phase mismatches, suppressing the growth of both pump harmonics and sidebands. Simultaneously, parametric amplification is phase-matched, so the signal increases nearly exponentially while propagating, leading to more than 20 dB of signal gain. All parameters as stated in the caption of Fig. 3.4.

to exponential, as it propagates along the TWPA. In this way, proper dispersion engineering enables the suppression of parasitic modes to nearly negligible levels, while simultaneously achieving parametric amplification close to the ideal three-mode behavior.

Wavenumber of a transmission line loaded with phase-matching resonators

In the presence of dispersion engineering, fine-tuning the pump frequency has a strong influence on the signal gain profile. To thoroughly investigate this, we first obtain an expression for the dispersion relation $k(\omega)$ modified by RPM. This can then be used for an approximate analytical description using the three-mode model of Sec. 2.4. The phase alteration by the RPM resonators can be approximately calculated using the transmission coefficient of a hanger-type resonator two-port (brown-shaded panels in Fig. 3.3) [114],

$$S_{21}^r = 1 - \frac{Q_L/Q_e}{1 + 2jQ_L(f - f_r)/f_r}, \quad (3.15)$$

which is an approximation in the vicinity of the resonance frequency $f_r = \omega_r/2\pi$,

$$\omega_r \approx \frac{1}{\sqrt{L_r(C_r + C_c)}}. \quad (3.16)$$

The linewidth of the resonance is determined by the loaded quality factor Q_L , which relates to the external and internal quality factors, Q_e and Q_i , via $Q_L^{-1} = Q_e^{-1} + Q_i^{-1}$. The internal quality factor represents internal losses for example in the dielectric, and the external quality factor represents loss due to coupling by the capacitor C_c to the external circuitry. The coupling capacitance C_c presents a degree of freedom in the circuit architecture, which turns out later as a useful parameter to make the circuit tolerant to fabrication spread of the resonators. The resonator two-port is loaded on both sides by the transmission line with impedance Z . After transforming the loaded resonator to an equivalent parallel circuit, similar to the approach in Ref. [115], the effective load resistance and effective coupling capacitance are

$$R_{L,\text{eff}} = \frac{1 + (\omega_r C_c Z/2)^2}{\omega_r^2 C_c^2 Z/2}, \quad (3.17)$$

$$C_{c,\text{eff}} = \frac{C_c}{1 + (\omega_r C_c Z/2)^2}, \quad (3.18)$$

and the external quality factor is

$$Q_e = \omega_r R_{L,\text{eff}}(C_r + C_{c,\text{eff}}). \quad (3.19)$$

Neglecting reflections by impedance mismatches, the phase shift of the resonator-loaded transmission line is composed of that by the (unloaded) unit cells, k_{unloaded} , given by the dispersion relation $k(\omega)$ of the homogeneous transmission line, Eq. 2.47, and that of the resonator two-ports. The circuit with N unit cells includes N/m resonator two-ports, each shifting the phase of a propagating wave by $\arg(S_{21}^r)$, so the average (loaded) phase shift per unit cell is

$$k_{\text{loaded}} \approx k_{\text{unloaded}} - \frac{\arg(S_{21}^r)}{m}. \quad (3.20)$$

This approximation was found to be in good agreement with small-signal simulations, with one example shown in Fig. 3.6 for the circuit parameters given in Tab. 3.2. A more rigorous analytical treatment is possible using the cascaded transfer matrix method (Sec. 3.2.3).

Choice of pump frequency in RPM

The dispersive RPM feature allows fine-tuning of the pump frequency to achieve phase matching of a particular triplet $\{f_p, f_s, f_i\}$, and near-phase-matched conditions in the vicinity of f_s . Considering phase matching in the center of the gain profile, i.e., at $f_s \approx f_i \approx f_p/2$, the required wavenumber of the pump is $k_p \approx 2k_s$ to achieve $\Delta k = 0$ (black dashed line). The phase shifts $\delta k_p = k_p^{\text{unloaded}} - k_p^{\text{loaded}}$ relative to a case without RPM (gray dashed line) are illustrated in Fig. 3.6 for four choices of f_p . The pump frequency marked in green roughly fulfills $\Delta k = 0$.

The resulting gain profile is depicted in Fig. 3.7, along with the gain profiles of three other choices of f_p , marked by same colors in Fig. 3.6. Note the increased relative bandwidth of the purple compared to the orange profile, which arises from perfect phase matching at frequencies far outside the center of the profile, while the orange profile does not exhibit perfect phase matching. In this way, fine-tuning f_p is a degree of freedom to influence the magnitude and the shape of the gain profile.

Gain profiles are simulated in WRspice (colored dots), with theory overlays (black dashed lines). The latter assume parametric amplification as an ideal three-mode process, Eq. 2.106, neglecting all parasitic modes, as justified above. Further, the theory curves use the phase-mismatch profiles $\Delta k(f_s, f_p)$, also plotted in Fig. 3.6, corresponding to the four choices of f_p and calculated using Eq. 3.20. Precisely, the gain profiles were calculated with identical circuit parameters, frequencies, and pump amplitude as used in the WRspice simulations, except the phase shift δk_p , which was taken as a free parameter to match visually the calculation to the shape of the simulated gain profiles. The resulting values of k_p are marked as colored dots in Fig. 3.6 and show only small deviations from $k(\omega)$ (brown line). Deviations are attributed to residual parasitic sideband modes beyond the three-mode model,

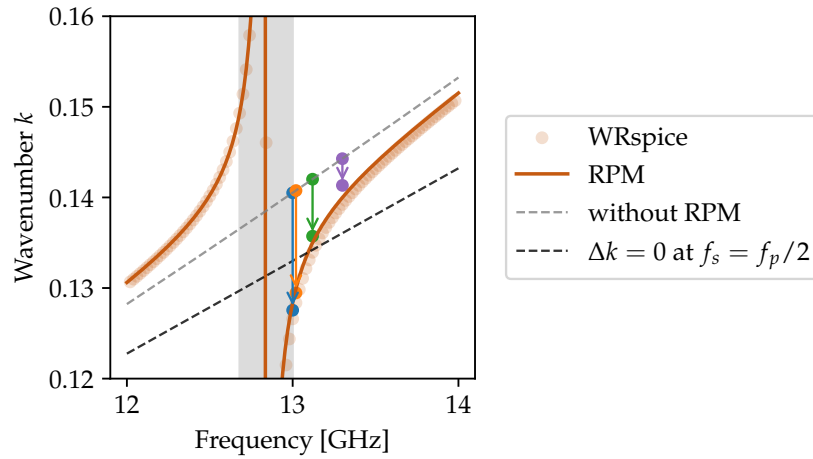


Fig. 3.6.: Phase matching by a nonmonotonic dispersive feature (RPM). Dispersion causes a phase mismatch $\Delta k = k_p - k_s - k_i$, equal to the vertical distance between the dashed lines for $f_s = f_i = f_p/2$ in the absence of RPM. The dispersive feature (brown line, Eq. 3.20) allows lowering k_p to compensate Δk by adequate choice of f_p (green arrow). The gray-shaded area marks the transmission dip, $|S_{21}| < 1$ dB. Brown circles depict a WRspice small-signal simulation. Circuit parameters see Tab. 3.2.

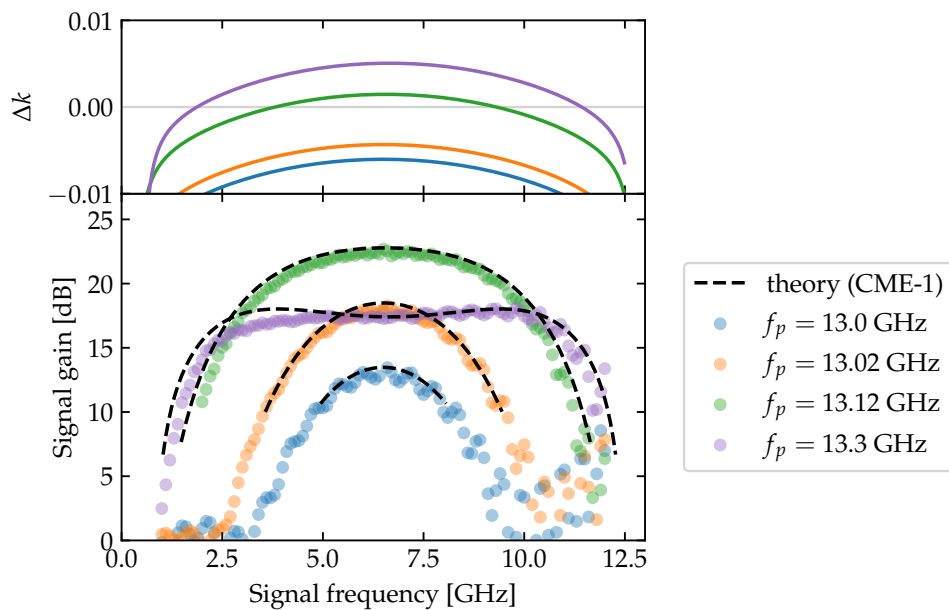


Fig. 3.7.: Phase mismatch Δk and signal gain of a 3WM TWPA with RPM. The pump frequency is a degree of freedom to change $\Delta k(f_s)$ (pump frequencies are marked as colored circles in Fig. 3.6). Bottom: Resulting gain profiles, simulated with WRspice. The green graph features high gain due to small Δk in a wide range of signal frequencies. The overlaid theory curves (dashed) indicate that parametric amplification is described well by a three-mode model due to strong suppression of other modes.

as well as to the unaccounted insertion loss experienced by the pump wave at frequencies in close vicinity to ω_r (gray-shaded area in Fig. 3.6).

In summary, a dispersion-engineering approach for 3WM TWPAs has been studied utilizing both circuit simulations and analytical modeling. While decreasing the plasma frequency is a simple and well-suited approach to suppress undesired parametric processes, the resonant phase-matching technique is a candidate for compensating the residual phase mismatch of parametric amplification. The combination of both methods enables high and broadband signal gains.

3.2.3 Periodic capacitance variation

As an alternative to RPM, another dispersion-engineering scheme was implemented for the purpose of phase matching, the technique called periodic capacitance variation (PCV). While RPM relies on inserting additional elements, the resonator-based phase shifters, to the circuit, PCV does not modify the circuit architecture, but utilizes a variation of the circuit parameters only. This potentially makes a practical implementation easier. In our approach, the ground capacitors C_n are varied periodically,

$$C_n = C_0 (1 + \zeta \sin(2\pi n/m)), \quad (3.21)$$

with C_0 the mean value, ζ the variation depth, and m the period, see Fig. 3.8. Similar to the RPM circuit in Figs. 3.3 and 3.4, dispersion is increased by an enlarged junction capacitance C_J for the purpose of mismatching unwanted parametric processes with frequencies above f_p . To calculate the dispersion relation resulting from PCV, the cascaded transfer matrix method is used.

Cascaded transfer matrix formalism

The formalism of cascaded transfer matrices (CTM) is a standard approach in optics [116] and in microwave engineering [85] to analytically describe arrangements of cascaded linear two-ports. Throughout this thesis, cascaded transfer matrices are used to calculate the dispersion relation and the scattering parameters of complicated TWPA circuits and passive test chips for experimental circuit parameter extraction (see Sec. 4.2.2), yielding remarkable agreement to experimental data (see Sec. 4.2.3).

Here, we employ this method to analyze the PCV-TWPA in the linear regime, i.e., neglecting nonlinearity by using the small-signal inductance L of the rf-SQUIDs (Eq. 2.24), valid for small amplitudes. Note that the single-tone transmission characteristics can deviate for large amplitudes (further analyzed in Sec. 3.3.3). The circuit comprises a cascade of N two-ports (unit cells), as depicted in the blue-

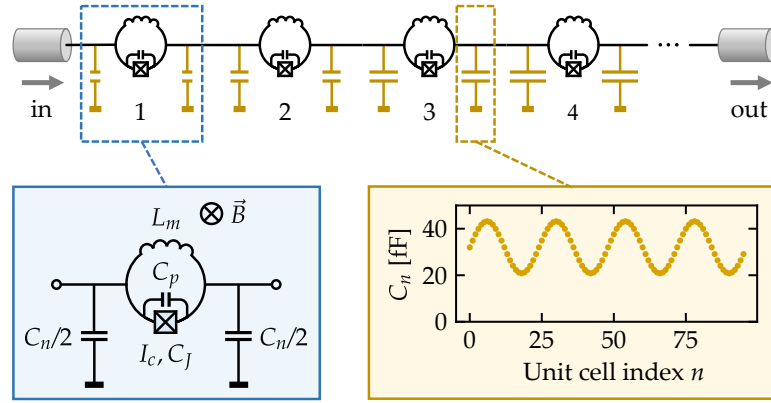


Fig. 3.8.: Scheme of a TWPA with periodic capacitance variation (PCV). The values of the ground capacitors C_n are periodically varied according to the scheme shown in the gold-colored panel. This creates a stopband in $k(\omega)$, used for phase matching of parametric amplification.

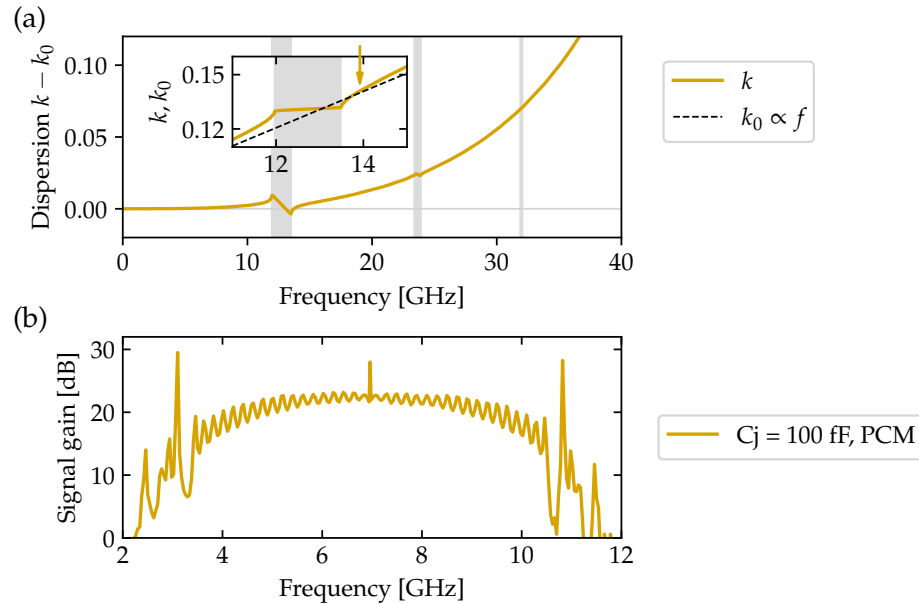


Fig. 3.9.: Wavenumber and signal gain in a TWPA with periodic capacitance variation (PCV). **(a)** Dispersive part of the wavenumber, $k - k_0$, where $k_0 = \omega/\omega_0$ is the linear dispersion. Low plasma frequency leads to strong dispersion of $k(\omega)$. Stopbands (gray-shaded areas) are created by PCV. The first stopband (inset) is used for phase-matching parametric amplification. **(b)** Simulated signal gain profile for the pump frequency marked by an arrow in the inset in (a). Increased ripple is a side effect of PCV. Peaks at the edges of the gain profile stem from phase matching of higher-order processes at the second stopband. Circuit parameters see Table 3.2. Simulations were performed with a pump current of $4.12 \mu\text{A}$, and a pump frequency of 13.92 GHz , a bias current of $7.5 \mu\text{A}$, and a signal current of $0.01 \mu\text{A}$.

shaded panel in Fig. 3.8. Each two-port n is characterized by a transfer matrix T_n (also known as the ABCD matrix),

$$T_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}, \quad (3.22)$$

with the coefficients

$$a_n = 1 - \frac{1}{2} \frac{\omega^2 L C_n}{1 - \omega^2 C_J L}, \quad (3.23)$$

$$b_n = \frac{i\omega L}{1 - \omega^2 C_J L}, \quad (3.24)$$

$$c_n = i\omega C_n - \frac{1}{4} \frac{i\omega^3 L C_n^2}{1 - \omega^2 C_J L}, \quad (3.25)$$

$$d_n = a_n, \quad (3.26)$$

where capacitances C_n are varied according to Eq. 3.21. These N transfer matrices T_n are then cascaded to a system transfer matrix

$$T_N = \begin{pmatrix} a_N & b_N \\ c_N & d_N \end{pmatrix} = \prod_{n=1}^N T_n = \left(\prod_{n=1}^m T_n \right)^{\frac{N}{m}} \quad (3.27)$$

taking into account the period m , assuming that N/m is an integer. The system transfer matrix can then be transformed to a scattering matrix [85] for a given reference impedance Z_0 . The input-reflection coefficient S_{11} and forward-transmission coefficient S_{21} are given by the expressions

$$S_{11} = \frac{a_N + b_N / Z_0 - c_N Z_0 - d_N}{a_N + b_N / Z_0 + c_N Z_0 + d_N}, \quad (3.28)$$

$$S_{21} = \frac{2}{a_N + b_N / Z_0 + c_N Z_0 + d_N}. \quad (3.29)$$

Because this circuit is analyzed in the linear (not pumped) regime, it is reciprocal, $S_{12} = S_{21}$, and because it is symmetrical, $S_{22} = S_{11}$. The dispersion relation can be calculated via the phase of the transmission coefficient,

$$k(\omega) = -\frac{\arg(S_{21})}{N}, \quad (3.30)$$

and is an average wavenumber. Note that the local wavenumber, which is the phase shift between adjacent unit cells, varies substantially on a sub-wavelength scale. For a parametric process, however, amplification or deamplification occurs on a length scale determined by the corresponding coherence length $\xi = \pi / \Delta k$, which is on a

scale of at least tens or hundreds of unit cells for the most relevant processes. Hence, the average wavenumber is the relevant quantity to express phase mismatches between copropagating waves. In essence, the discrete circuit can be regarded as a metamaterial. This metamaterial behaves similarly to a continuous medium, but its macroscopic properties are determined by discrete elements on a sub-wavelength scale.

Band structure of a PCV-TWPA

The periodic variation of the capacitances leads to a corresponding periodic variation of the local phase velocity, $v_{\text{ph},n} = a / \sqrt{LC_n}$. Its shape is similar to the Kronig-Penney model [117], describing the behaviour of electrons in a periodic potential and leading to the band structures of periodic crystalline materials. Analogous to bandgaps in periodic crystals, a bandgap structure appears in the dispersion relation $\omega(k)$ of a periodic one-dimensional metamaterial such as the PCV-TWPA. Due to this analogy, TWPAs with periodic circuit parameter variation are sometimes called photonic-crystal TWPAs [50].

The resulting band structure is given by the dispersion relation $\omega(k)$, which can be obtained by Eq. 3.30. It is plotted in Fig. 3.9a, where for better visibility of the dispersive effects the linear wavenumber $k_0 = \omega / \omega_0$ is subtracted from $k(\omega)$. The band structure contains passbands, where waves can propagate, and stopbands (bandgaps), where wave propagation is forbidden. Stopbands occur at wavenumbers $k_j = jk_m / 2$, with index $j = 1, 2, 3, \dots$ and the crystal wavenumber $k_m = 2\pi / m$. At these wavenumbers, the phase shift across the period m is $j\pi$ which, for $j = 1$, is equivalent to the condition for the cutoff in a discrete transmission line, see Sec. 2.2.2. Three such stopbands can be seen in Fig. 3.9a, but only the first is clearly pronounced. Stopbands are centered approximately around $\omega_j = (j\pi / m)\omega_0$, ignoring the dispersive effect of ω_j .

Phase matching of parametric amplification is enabled by an appropriate choice of f_p such, that the phase mismatch is compensated by the dispersive shift in the vicinity of the stopband. For $f_s = f_i = f_p / 2$, this condition is met when the pump frequency satisfies $k(f_p) = 2k_s \approx 2f_s / f_0 = f_p / f_0$, as illustrated in the inset of Fig. 3.9a. The PCV variation depth ζ is chosen large enough that such a point exists.

Gain profile of a PCV-TWPA

The combination of a low plasma frequency and the first stopband created by PCV effectively suppresses unwanted parametric processes, while simultaneously providing phase matching for parametric amplification, as described in Sec. 3.2.2.

Therefore, the parametric processes are again close to the ideal three-mode case. Figure 3.9b shows the simulated signal gain profile. The gain and bandwidth are comparable to that of the RPM-TWPA, but the profile exhibits significantly higher gain ripple. This ripple is a side effect of PCV and are discussed in detail in Sec. 3.3. Noteworthy are the pronounced peaks and broad gain dips at both edges of the gain profile. At these frequencies, the sideband frequencies f_{p+s} or f_{p+i} , involved in parametric up-conversion processes, coincide with the second stopband. Similar behaviour has been observed experimentally, see Figures 4.17 and 4.18 in Sec. 4.5.

3.2.4 Multiperiodic capacitance variation

In a third variant of dispersion engineering for 3WM TWPAs, strong dispersion, necessary for suppressing undesired parametric processes, is achieved through a wide stopband rather than by a low plasma frequency. An additional narrow stopband enables phase matching of the intended parametric amplification. This approach, called multiperiodic capacitance variation (mPCV), varies the ground capacitors C_n in a multiperiodic scheme. This chapter discusses a simulation study of the properties of such a TWPA, published in part in Ref. [94]. The experimental implementation of an mPCV-TWPA, along with a detailed experimental study, is presented in Chapter 4.

Circuit design and band structure of an mPCV-TWPA

Our circuit is designed such that the first stopband (index $j = 1$) in the resulting band structure $\omega(k)$ is narrow, while the second stopband ($j = 2$) is wide. This is achieved by varying the ground capacitances C_n with three discrete capacitance values C_{01}, C_{02} and C_{03} , as illustrated in Fig. 3.10a. This modified capacitance profile has two variation depths,

$$\zeta_1 = \frac{C_{02} - C_{03}}{C_{02} + C_{03}}, \quad (3.31)$$

$$\zeta_2 = \frac{C_{02} + C_{03} - 2C_{01}}{C_{02} + C_{03} + 2C_{01}}, \quad (3.32)$$

where ζ_1 is the variation depth corresponding to the fundamental period m (or spatial frequency of $k_m/(2\pi) = 1/m$), and ζ_2 corresponds to the half-period variation $m/2$ (or double-frequency $2/m$). Crucially, the variation depth of the double-frequency component $2/m$ is significantly larger than that of the fundamental spatial frequency, $1/m$. Consequently, the width of the second gap in the dispersion relation $k(\omega)$ is notably larger than that of the first gap. The shape of $k(\omega)$ can be calculated using the cascaded transfer matrix formalism, Eqs. 3.26–3.30, and is

shown in Fig. 3.10b. It exhibits a narrow first and wide second stopband. Also visible is a third, higher-order, stopband.

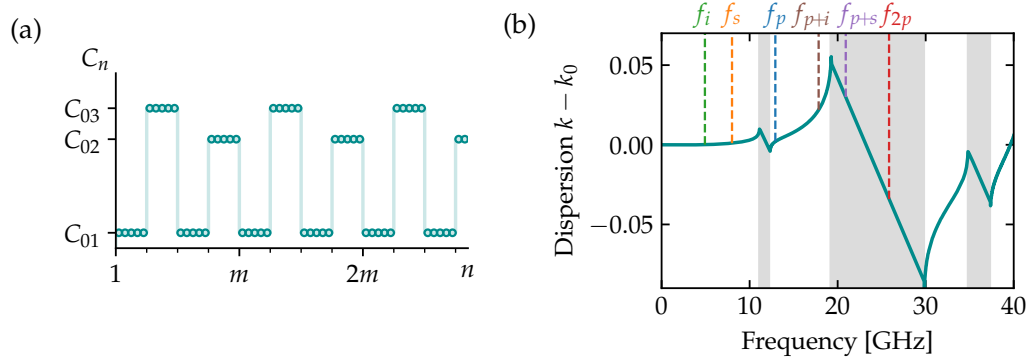


Fig. 3.10.: Dispersion engineering via multiperiodic capacitance variation. **(a)** Scheme showing multiperiodic variation of ground capacitances C_n with fundamental period m and half-period $m/2$. The capacitances C_n with index $n = 1 \dots N$ are varied between three constant values. In the scheme, $m = 20$. **(b)** Resulting dispersion relation. Plotted is the dispersive part $k - k_0$, with $k_0 = \omega/\omega_0$. Gray-shaded areas denote stopbands, where waves cannot propagate. The first stopband corresponds to variation period m , and the second to half-period $m/2$. Vertical dashed lines illustrate phase matching of pump, signal, and idler, and suppression of the unintended 3WM products f_{p+s}, f_{p+i}, f_{2p} , for the example $f_s = 8$ GHz and $f_p = 12.92$ GHz. Phase matching is achieved by lowering k_p at the first stopband, while unwanted tones are suppressed either by phase mismatch (as with f_{p+i}), or by falling into the second stopband, inhibiting propagation (f_{p+s} and f_{2p}). Circuit parameters see Table 3.2.

Phase-mismatching by inhibiting propagation

Phase matching for parametric amplification is achieved, in the same way as for PCV, by placing the pump frequency f_p slightly above the first stopband to lower k_p (Fig. 3.10b). Undesired tones, e.g., f_{p+s}, f_{p+i}, f_{2p} , are either phase-mismatched by strong dispersion, similar to the case with low plasma frequency, or fall into the wide stopband, where propagation is inhibited. In the latter case, these tones are most effectively suppressed. Figure 3.11 shows the spatial evolution of the six most relevant tones. The multimode dynamics are similar to those in the TWPA concepts employing phase-mismatching via low plasma frequency (Fig 3.5c), but in comparison, unwanted modes are suppressed even stronger in the mPCV-TWPA. For example, the second harmonic of the pump is ca. 20 dB lower in amplitude than the pump (compared to 12 dB) and the sideband tone f_{p+s} is ca. 10 dB lower than signal and idler (compared to 5 dB). Therefore, parametric amplification is even closer to the ideal three-mode model described in Sec. 2.4.3.

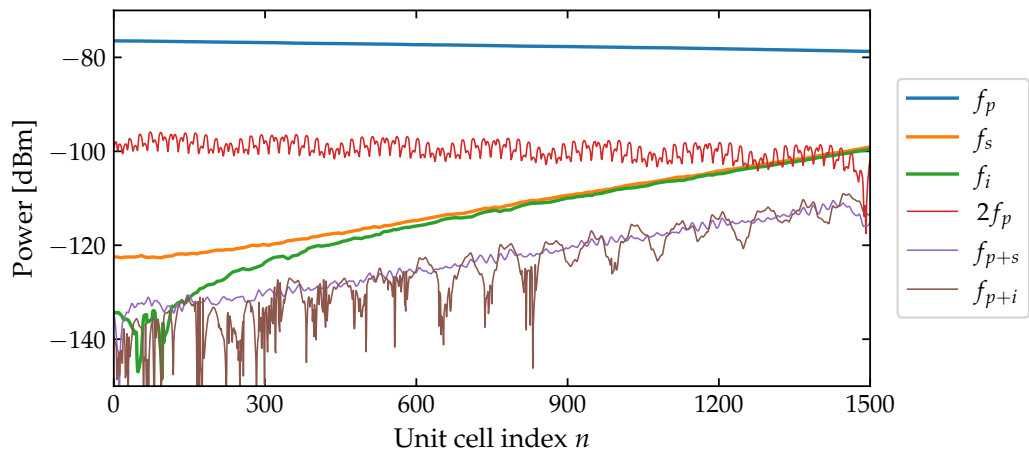


Fig. 3.11.: Evolution of the mode powers of the six most relevant waves along the TWPA. While the mode f_{p+i} is suppressed by phase-mismatch, similar to the the RPM-TWPA of Fig. 3.5c, the modes f_{2p} and f_{p+s} fall into the second stopband and cannot propagate. The signal and idler waves increase nearly exponentially along the circuit. Circuit parameters see Tab. 3.2. Simulation parameters are $I_p = 2.0 \mu\text{A}$, $f_p = 12.92 \text{ GHz}$, $I_s = 0.01 \mu\text{A}$, and $f_s = 6.7 \text{ GHz}$.

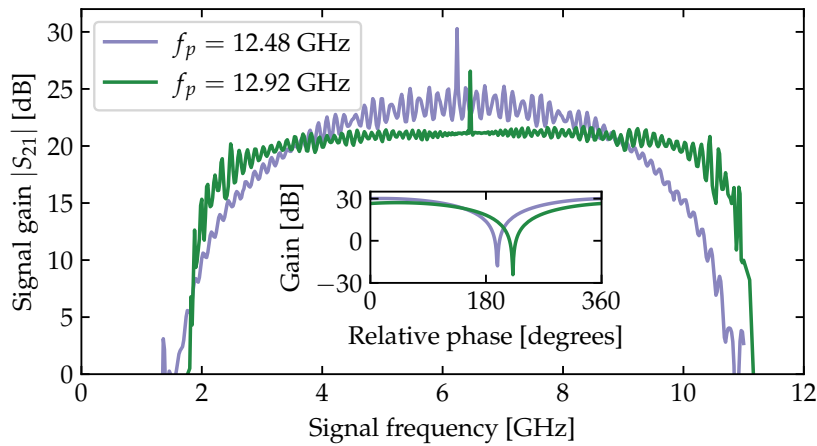


Fig. 3.12.: Signal gain spectrum of an mPCV-TWPA. The choice of pump frequencies allows tuning the shape of the gain profile, c.f. Fig. 3.7. The peaks in the center of the curves occur due to degenerate 3WM, i.e., $f_s = f_i = f_p/2$, where parametric amplification is phase-sensitive, as shown in the inset. Here, the signal phase was tuned relative to the phase of the pump. Phase-sensitive gain varies periodically between amplification in one quadrature and deamplification in the other. The maximum degenerate gain is ca. 6 dB higher than the nondegenerate gain in the vicinity of $f_p/2$, while the maximum extinction ratio is ca. 48 dB and 51 dB for the two pump frequencies, respectively. The pump amplitude was $I_p = 1.8 \mu\text{A}$ in these simulations. Circuit parameters see caption of Fig. 3.11, except pump amplitude $I_p = 1.8 \mu\text{A}$.

The reason for their strong suppression is that these modes, although generated along the circuit, effectively cannot propagate, so they are hindered in growth. The distinct beating pattern of the second harmonic f_{2p} exhibits an envelope period of $\pi/(2k_p - k_m) = 104$ unit cells, indicating the coherence length of the second-harmonic generation process, and a superposed pattern of $m = 20$ unit cells, indicating the propagation-inhibiting interferences.

Gain profile

The corresponding gain profile is presented in Fig. 3.12. It shows two examples of continuous broadband gain, found for pump frequencies in a range of ca. 12.4–13.2 GHz. Similar to Fig. 3.7, the pump frequency can be fine-tuned to influence the shape of the gain profile. While at $f_p = 12.48$ GHz higher gain is achieved, the slightly higher pump frequency of 12.92 GHz leads to remarkably broadband flat gain. In the former case, phase matching is achieved around the center of the gain profile, whereas in the latter case, phases are matched towards the edges of the gain profile, resulting in a flat and wide profile.

The achievable bandwidth is limited by the width of the first stopband in two ways. First, the gain drastically drops when either the signal or the idler frequency approaches the lower edge of the stopband, where propagation is inhibited. Second, the bandwidth is a function of the convexity of the dispersion relation [118], determining $\Delta k(f_s)$. In some cases, it may be beneficial to deliberately reduce the (inherently wide) bandwidth of the amplifier, for example to match it to the limited bandwidth of circulators or isolators. This can be done by increasing the width of the stopband via increasing variation depth ζ_1 . Note that increased gain ripple may be a side effect of wide stopbands, as analyzed in more detail in Sec. 3.3.

Phase-sensitive and phase-preserving parametric amplification

A distinct feature in both gain profiles in Fig. 3.12 (dito in Fig. 3.4) are the peaks at $f_s = f_p/2$. These peaks emerge due to degenerate 3WM parametric amplification, where the signal and idler frequencies coincide, $f_i = f_s$ (see Sec. 2.4.4). In parametric amplification, the phase of the idler is determined by the phases of signal and pump. In the degenerate case, the phases of signal and idler can interfere constructively (amplification) or destructively (leading to deamplification), depending on the phase of the signal relative to that of the pump. For the simulation shown in the inset of Fig. 3.12, the phase of the signal current, injected into the circuit, was varied relative to the phase of the pump current. The resulting maximum gain is 6 dB higher than the phase-insensitive gain in the vicinity of $f_p/2$. The ratio between

maximum amplification and maximum deamplification is called extinction ratio and is ca. 50 dB in the simulation.

The phase-sensitive regime is particularly interesting due to the possibility of single-mode squeezing, enabling amplification of one quadrature of a signal with added noise lower than the quantum limit of phase-insensitive amplification [5, 35, 36]. Squeezing has been demonstrated by Perelshtein et al. in a SNAIL-based 3WM TWPA [76]. In a 3WM TWPA, this regime is easily accessible, because the pump frequency is well separated from the signal and idler frequencies, $f_p = 2f_s = 2f_i$. This is not the case for conventional 4WM TWPAs, where $f_p = f_s = f_i$ in the doubly-degenerate case (doubly-degenerate refers to four waves all having the same frequency). However, Qiu et al. recently demonstrated phase-sensitive amplification and squeezing in a 4WM TWPA using two (nondegenerate) pumps, $f_{p1} < f_s = f_i < f_{p2}$, and reported a large extinction ratio of 56 dB [90].

3.3 Reflections in TWPAs

The strong gain ripple observed especially in Figures 3.9b and 3.12 is an unwanted but expected symptom of reflections inside the TWPA or in its embedding circuitry [38, 50, 69, 119]. Due to their similarity to multireflections in optical cavities, they are sometimes referred to as Fabry-Perot interferences. Microwaves reflected from a TWPA can have higher amplitudes than the incident waves, presenting a potentially serious issue. When for example the tone used to dispersively readout the state of a qubit travels back to the qubit with higher amplitude, it can cause state transitions out of the computational subspace of the qubit [29]. To avoid such issues, high isolation is required between TWPA and qubit chip. This is typically achieved by one or several isolators, see e.g. Refs. [65, 120]. Since these devices are bulky, costly, lossy, and magnetic, they are a limiting factor in scaling up quantum computers. Therefore, reducing reflections in TWPAs is a crucial task, and the search either for alternatives to ferrite-based isolators [121, 122] or for truly nonreciprocal TWPAs, ideally having high gain for forward-propagating and high isolation for backward-propagating signals, is an emerging field of research [123, 124]. In this section, we analyze reflections, gain ripple, and stability in TWPAs on the example of the mPCV-TWPA described in the preceding section. While especially pronounced for TWPAs with periodic circuit parameter variation, reflections and gain ripple are inherent to all TWPAs without backward isolation, so the results are applicable to other TWPA architectures than the mPCV-TWPA.

3.3.1 Gain ripple and instability by signal feedback

We first analyze with a simple model the consequences of signal reflections in a TWPA, which is assumed to be slightly impedance-mismatched to its embedding environment (Fig. 3.13a). In this model, idler reflections are ignored. The characteristic impedance $Z(f_s)$ of the TWPA is in general a function of the signal frequency f_s , given for example by Eq. 2.44. It is close to but not equal to the source impedance Z_S and the load impedance Z_L , defining the voltage reflection coefficients

$$\Gamma_L = \frac{Z_L - Z}{Z_L + Z}, \quad (3.33)$$

$$\Gamma_S = \frac{Z_S - Z}{Z_S + Z}, \quad (3.34)$$

seen from a wave propagating forward or backward along the TWPA, respectively. All impedances are assumed real-valued. For simplicity, we neglect signal reflections at the TWPA input back to the source, given by a reflection coefficient Γ_{in} ,

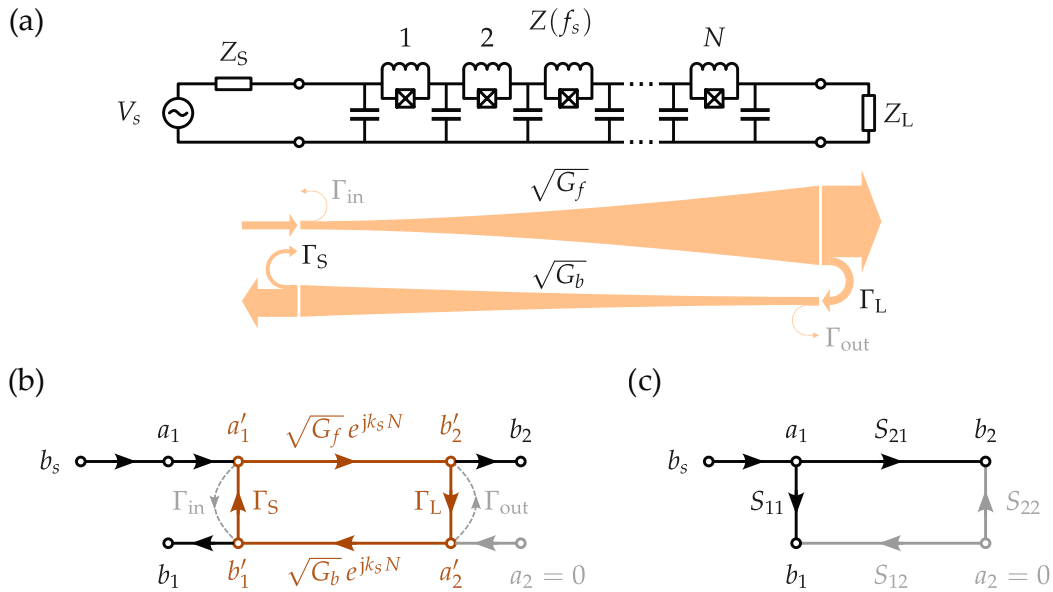


Fig. 3.13.: Signal reflections in a TWPA. (a) Illustration of a TWPA with impedance $Z(f_s)$, mismatched to the load and source impedances. The amplified forward-propagating signal is reflected at the output. Reflected pump amplifies backward-propagating signal waves with backward power gain G_b . Signal reflections (reflection coefficients Γ_S, Γ_L) form a feedback loop. (b) Representation of the circuit as a microwave signal-flow chart, exhibiting a self-loop (brown) with roundtrip power gain $G_{loop} = \Gamma_S^2 \Gamma_L^2 G_f G_b$ and electrical length of $2k_s N$. The loop causes gain ripple and potential instability. The incident signal power wave is $b_s = \sqrt{Z_0} V_s / (Z_S + Z_0) \approx V_s / 2\sqrt{Z}$ for $Z \approx Z_S$. (c) Decomposition of (b) to a loop-free chart. By taking $Z_S = Z_L = Z_0$ as reference impedance, the branches denote the scattering parameters of the TWPA.

and reflections at the output, Γ_{out} , back to the load. As Fig. 3.13a makes plausible, these reflections play only a minor role. In the TWPA, forward-propagating signals are amplified by a forward power gain G_f , while backward-propagating signals are amplified by a backward power gain G_b . The latter, further investigated in Sec. 3.3.2, arises mostly due to reflections of the pump wave, but it can also emerge because of nonlinear forward-backward coupling by other parametric processes [59]. While propagating along the circuit, the signal wave accumulates a phase shift of $k_s N$.

Signal reflections in the TWPA form a feedback loop, as illustrated in Fig. 3.13a. This loop is the cause of gain ripple and instabilities. To obtain expressions for the resulting scattering parameters, and to examine stability, we utilize the signal-flow-chart approach from microwave theory [85]. The signal-flow chart corresponding to the circuit is depicted in Fig. 3.13b. It contains signal power-wave quantities on the nodes and scattering parameters on the branches. The responses of the TWPA to an injected signal with incident power wave $a_1 \equiv b_s = \sqrt{Z_0} V_s / (Z_S + Z_0)$ are

the reflected and transmitted power waves b_1 and b_2 . These define the scattering parameters $S_{11} = b_1/a_1$ and $S_{21} = b_2/a_1$ (Fig. 3.13c), given that $a_2 = 0$, and for a reference impedance which we set as $Z_S = Z_L = Z_0$ for convenience. To derive the scattering parameters, the reflections causing the feedback loop are treated as being internal to the TWPA (brown loop in Fig. 3.13b). A signal wave propagating around the loop accumulates a phase difference of $2k_s N$ and is amplified by

$$\sqrt{G_{\text{loop}}} = \Gamma_L \Gamma_S \sqrt{G_f G_b} \quad (3.35)$$

in one roundtrip. The loop can be resolved via flow-chart decomposition rules [85] to find expressions for $b_1 \equiv b'_1$ and $b_2 \equiv b'_2$. First, resolving the self-loop using Mason's Rule yields $a'_1 = a_1 / (1 - \sqrt{G_{\text{loop}}} e^{2jk_s N})$. Then, combining this with the path $a'_1 \rightarrow b'_2$, the flow chart reduces to a single branch from a_1 to b_2 with the forward-transmission coefficient

$$S_{21} = \frac{b_2}{a_1} = \frac{\sqrt{G_f} e^{jk_s N}}{1 - \Gamma_L \Gamma_S \sqrt{G_f G_b} e^{2jk_s N}}. \quad (3.36)$$

Elegantly, the same equation was found as the convergence of an infinite geometrical series of multireflections, as done in Ref. [50]. However, the signal-flow-chart approach, common in microwave engineering, may be easier to extend for more complicated schemes. Similar to S_{21} , we get the input-reflection coefficient by extending the path to $a'_1 \rightarrow b'_1$,

$$S_{11} = \frac{b_1}{a_1} = \frac{\Gamma_L \sqrt{G_f G_b} e^{2jk_s N}}{1 - \Gamma_L \Gamma_S \sqrt{G_f G_b} e^{2jk_s N}}. \quad (3.37)$$

Gain ripple and amplified signal reflections

The scattering parameters of a TWPA with and without considering signal feedback are depicted in Fig. 3.14. The TWPA in the example has unity backward gain, $G_b = 1$, and is reasonably well impedance-matched to its environment, $\Gamma_S = \Gamma_L = -20$ dB. Without signal feedback, the signal power gain, $G = |S_{21}|^2$, does not have ripples, and the signal input reflections are low, $S_{11} = \Gamma_L = \Gamma_S$.

On the other hand, taking into account signal feedback, the power gain exhibits notable ripple, similar to those of the WRspice simulation of the mPCV-TWPA for $f_p = 12.48$ GHz, shown in Fig. 3.12. These are caused by the denominator of Eq. 3.36, from which it is apparent that ripple increases with increasing gain. Adjacent ripples have a frequency spacing of $\Delta f = v_{ph}/2\ell \approx \omega_0/2N$, corresponding to twice the electrical length of the TWPA. In this simple model presented here, only signal reflections are accounted for. Realistically, the idler experiences reflections as well,

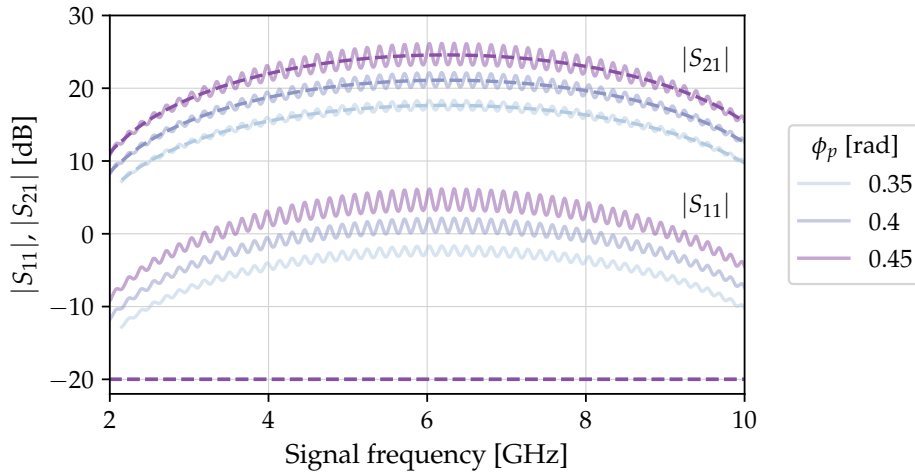


Fig. 3.14.: Fabry-Perot interferences in a TWPA. Transmission coefficient $|S_{21}|$ and reflection coefficient $|S_{11}|$ are depicted with (solid lines) and without (dashed lines) considering gain feedback from reflections at the input and output of the TWPA with $\Gamma_S = \Gamma_L = -20$ dB and $G_b = 0$ dB. Higher gain $|S_{21}|^2$ due to increased pump amplitude ϕ_p causes higher ripple and higher reflection. The ripple spacing corresponds to twice the electrical length of the TWPA. Reflected signal can be larger than the incident signal, $|S_{11}| > 0$ dB. Curves are calculated with the analytic three-mode model (Eq. 2.106) with circuit parameters of the mPCV-TWPA of Tab. 3.2 and phase mismatch calculated via CTM (Sec. 3.2.3) for $f_p = 12.46$ GHz.

which are nonlinearly coupled to the signal. Their possible destructive interference with the signal reflections is one reason for the lower gain ripple, having frequency spacing deviating from $\Delta f \approx \omega_0/2N$, as found for $f_p = 12.92$ GHz in Fig. 3.12.

Moreover, the reflected signal is larger than the incident signal, $|S_{11}| > 0$ dB, in wide frequency ranges. Reflected signal waves, propagating back to the signal source, can be a serious issue. For example, when a qubit readout tone reflects at a TWPA and propagates back to the qubit with higher amplitude, it can cause unintended population of higher qubit states [29]. To prevent this, isolators are typically placed between TWPA and qubit chip.

Instability and parametric oscillations

While gain ripple is a symptom of small reflections, feedback can generally cause instability in an amplifier [85]. While a detailed treatment of stability in the TWPA is beyond the scope of this thesis, we can get a simple stability criterion from the model described above, indicating some upper bounds. Omitting phase dependences, instability can occur when the loop gain, Eq. 3.35, is larger than unity [125]. Then, the signal traveling around the feedback loop increases in amplitude each roundtrip. The diverging amplitude is limited only by the available pump power driving

parametric amplification, which gets drastically depleted. Even if instability occurs only for a single frequency, the resulting pump depletion can severely limit the achievable signal gain for all other frequencies.

Interestingly, even in the absence of an injected signal, quantum fluctuations at the signal frequency can escalate to very high amplitudes at the frequencies where the criterion for instability is fulfilled, and cause pump depletion. This effect of diverging amplification of quantum fluctuations is also known as parametric oscillations [56]. In the TWPA presented in Chapter 4, parametric oscillations have been observed under some conditions, and are likely the upper bound to the achievable gain. Further, this effect has been utilized for halftone (subharmonic) generation in a superconducting resonator with embedded flux-biased rf-SQUID, generating a tone at half of the pump frequency via 3WM [126].

Practical considerations

In practice, it is challenging to achieve broadband impedance matching much better than -20 dB. Considering unity backward gain, and a forward gain of 20 dB, the amplifier is stable, because the resulting loop gain is -20 dB. In the presence of backward gain, however, or for worse impedance matching, instability can easily occur at a forward gain of 20 dB. This may be the case, for example, for a backward gain of 5 dB and impedance matching of -12.5 dB, values which are not unrealistically high (c.f. Fig. 4.8). Furthermore, it may not be the TWPA causing reflections, but its packaging, e.g. bond wires, or exterior circuitry, such as isolators. Particularly the latter have a limited bandwidth, outside of which they are often poorly impedance-matched. This is one factor why increased gain ripple is observed experimentally (c.f. Fig. 4.28). Note that it is therefore not without drawbacks to have a TWPA with a gain bandwidth exceeding that of the isolators.

3.3.2 Backward amplification

Ideally, a TWPA is a directional amplifier, amplifying in forward direction only, while the backward transmission is close to unity. Realistically, however, reflections of the pump wave are likely due to imperfect impedance matching at the pump frequency. The interaction of the reflected pump (wavenumber $-k_p$) with the reflected signal ($-k_s$) enables backward amplification [50, 69], following the same phase-matching criterion as the forward process, Eq. 2.76. In a TWPA with PCV, backward amplification is also possible by the pump wave component $k_p - k_m$ [59]. However, phase matching with backward-propagating signal and idler waves, $k_p - k_m + k_s + k_i \approx 0$, is usually only fulfilled in very narrow frequency ranges [50]. Since backward gain is a factor in the loop gain, Eq. 3.35, describing signal feedback,

it has an enhancing effect both on gain ripple and on the effective input reflection coefficient S_{11} . Therefore, backward gain is an unwanted effect.

To investigate backward amplification in simulations, the circuit model of the mPCV-TWPA (Sec. 3.2.4) is modified such that the signal source is connected to the nominal output of the TWPA, opposite of the pump source. Then, the signal wave, propagating in opposite direction to the pump wave, is amplified only if a sufficient part of the pump wave is reflected, or if nonlinear forward-backward coupling, such as that of the pump wave component $k_p - k_m$, is phase-matched. The signal gains probed in forward and backward direction, S_{21} and S_{12} , respectively, are shown in Fig. 3.15a for two different pump frequencies. Especially for $f_p = 12.46$ GHz, the high backward gain presumably plays a large role in the observed high ripple of the forward gain. Vice versa, the forward gain enhances gain ripple in the backward gain for the same reason.

Since the backward gain is rather broadband, exhibiting the same overall shape as the forward gain, it suggests that the reflected pump wave, $-k_p$, is the dominating source of backward amplification. Further insights are possible via the spatial Fourier transform of the pump waves, performed on the pump amplitudes on the N unit cells in the circuit. The spectra are shown in Fig. 3.15b. The main peak is the forward-propagating pump-wave component with wavenumber k_p , slightly above the crystal wavenumber $k_m/2$ (Sec. 3.2.3), because the pump frequency is fixed slightly above the first stopband. However, also the backward-propagating pump-wave components at wavenumbers $-k_p$ and $k_p - k_m$ have substantial amplitudes. While the former wave component is the reflection of the pump wave at the output of the TWPA, the latter stems from periodic multireflections inside the TWPA, which form the stopbands. In agreement with the observed backward amplification, the backward-propagating pump-wave components are larger at $f_p = 12.46$ GHz and lead to larger backward gain ripple. This indicates that impedance matching at these pump frequencies is rather poor.

3.3.3 Ripple despite nominal impedance match: Stopband sidelobes

Minimizing reflections via proper impedance matching at both the signal frequencies and the pump frequency are of paramount importance in achieving high gain with low ripple and low backaction in a TWPA. An advantage of SQUID-based TWPAs is the flux-tunability of their characteristic impedance. In the case of rf-SQUID-based TWPAs, this impedance is given by Eq. 2.44 and is approximately $Z = \sqrt{L(\Phi_{\text{ext}})/C_0}$ in homogeneous rf-SQUID transmission lines for sufficiently low frequencies $\omega \ll \omega_J, \omega_0$. This flux dependence is rather steep in the vicinity

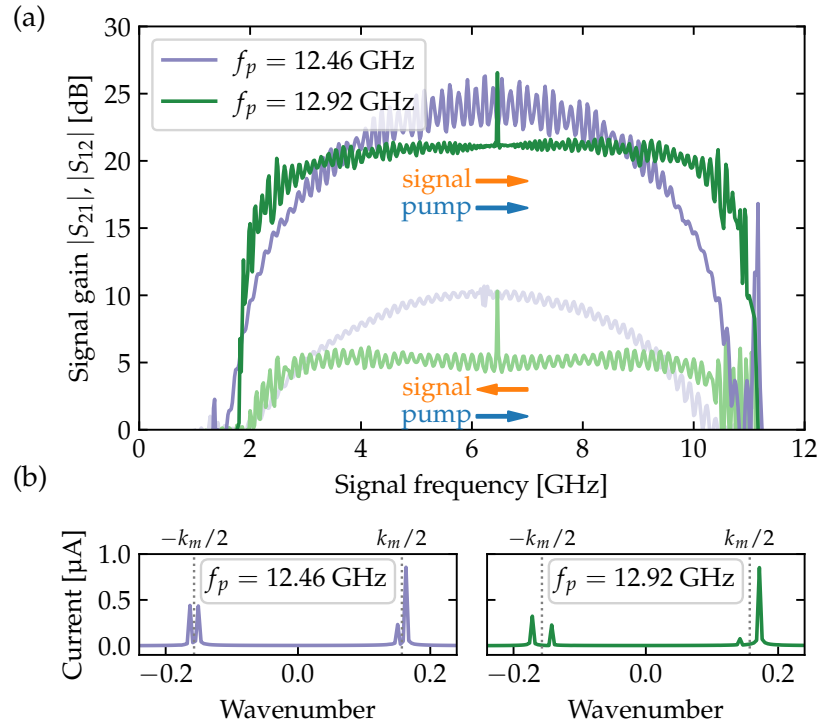


Fig. 3.15.: Amplification of backward-propagating signal waves by reflected pump. **(a)** Signal gain for forward (S_{21} , dark-colored) and backward (S_{12} , light-colored) traveling signal waves. To simulate S_{12} the signal and the pump are injected at opposite ends of the array. Due to Fabry-Perot-like interference, higher backward gain leads to higher ripple on the forward gain. Backward amplification is possible due to undesired backward-propagating pump-wave components (peaks centered around $-k_m/2$ in Panel (b)). Panel **(b)** shows the spatial Fourier transform of the pump waves of Panel (a) ($I_p = 1.8 \mu\text{A}$ in all simulations). The pump wave contains not only the forward-propagating component with k_p (main peak), but also a significant backward-propagating component $k_p - k_m$, as well as their reflected counterparts, $-k_p$ and $-k_p + k_m$, respectively. The latter three components are generally more pronounced for pump frequencies closer to the stopband. Component $-k_p$ is dominant for backward amplification.

of the envisaged operation point, where $\gamma \approx 0$ and β is close to its maximum (c.f. Fig. 2.4). Thus, in-situ fine-tuning the impedance by adjusting the magnetic flux or bias current is possible within some range without significantly changing the nonlinear behaviour of the circuit. In this way, deviations of the circuit parameters from their target values can be at least partially compensated.

Although the circuit model used for simulations does not include imperfections or deviations from target values, no bias current was found showing perfect impedance matching and full suppression of gain ripple. Instead, the residual impedance mismatch originates from the periodic impedance variation along the circuit due to periodic capacitance variation. In fact, the impedance $Z(\omega)$ exhibits a pattern nearly periodic in frequency, deviating from the average impedance $\bar{Z} = \sqrt{L/\bar{C}_n}$, where $\bar{C}_n$ denotes the average ground capacitance, only valid for low frequencies or long wavelengths $\lambda \gg m$. The same pattern can be observed, for example, in the reflection coefficient $S_{11}(f)$, shown in Fig. 3.16. Apart from the stopband around 12 GHz (the main lobe), a large number of sidelobes are visible, manifesting as frequency-periodic increased reflection and reduced transmission. These sidelobes exhibit reflection coefficients larger than -20 dB for typical signal frequencies in the band of 3–9 GHz, which causes notable gain ripple (compare Fig. 3.14).

In the range of typical pump frequencies, 12.4–13 GHz, the reflection coefficient is much larger (insets of Fig. 3.16). The pump frequencies used for Fig. 3.15 are marked as vertical dash-dotted lines, indicating the choice of minimized pump reflection and maximized transmission. Still, the reflection coefficient is rather large with ca. -6 dB at 12.46 GHz, explaining the observation of strong backward amplification in Fig. 3.15. Note that for increasing pump amplitudes the stopband as well as its sidebands slightly shift towards lower frequencies. This effect is attributed to residual Kerr nonlinearity and is much smaller than that observed in the 4WM photonic-gap TWPA of Ref. [50], where a theoretical and experimental analysis of this shift is offered.

We can conclude that sidelobes are the main origin of gain ripple when the average impedance is matched, $\bar{Z} = Z_0$. They are immanent to TWPAs with periodic circuit parameter variation or periodic loading, representing a conceptual disadvantage over other dispersion engineering concepts, such as RPM, compare Figures 3.9 and 3.4. For experimental studies on sidelobes of our devices refer to Sec. 4.2.3. Recently, calibrated measurements of the full scattering matrix of a photonic-gap TWPA has shown sidelobes causing reflection coefficients of up to -10 dB [119].

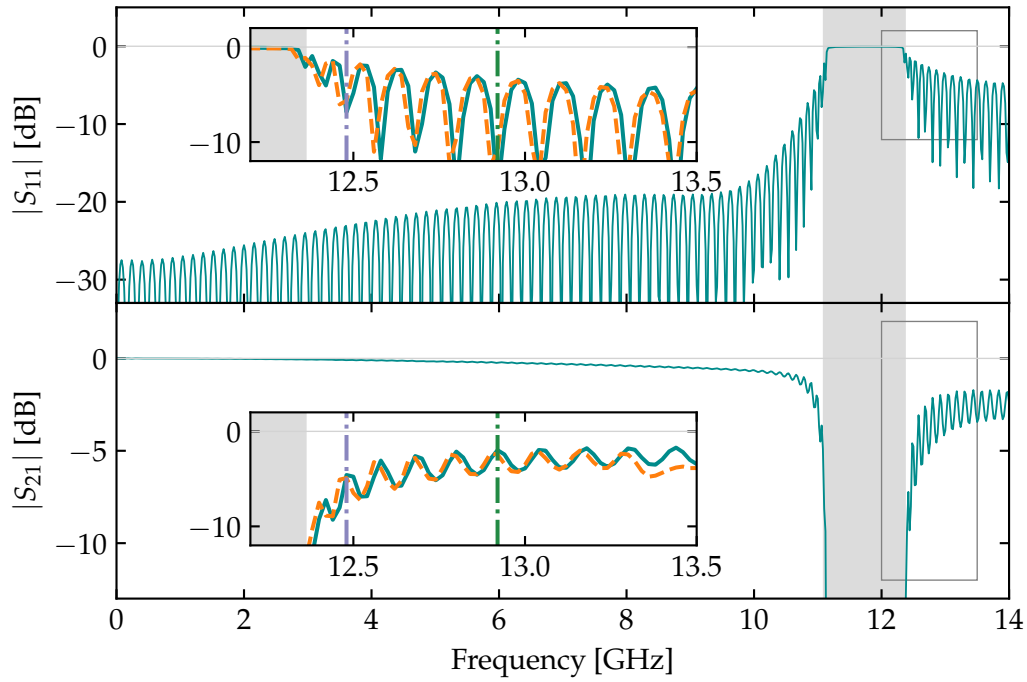


Fig. 3.16.: Reflection and transmission coefficients, S_{11} and S_{21} , for typical frequencies of signal, idler, and pump. The nearly frequency-periodic pattern apparent in S_{11} are the sidelobes of the first stopband (gray-shaded area). Insets show close-up views in the pump-frequency range slightly above the stopband. Green solid lines are in the small-signal regime ($I_p = 0.01 \mu\text{A}$), and dashed orange lines in a large-signal regime ($I_p = 1.8 \mu\text{A}$). For increasing pump powers, the stopband and its sidelobes shift to lower frequencies. Pump frequencies used in Fig. 3.15 are indicated by dash-dotted lines of same colors. Curves are obtained by WRspice simulations with only a pump injected ($I_s = 0$).

3.3.4 Sidelobe suppression by apodized capacitance variation

A way to eliminate or drastically reduce sidelobes is the apodization technique, widely used in fiber-optical Bragg gratings [127, 128]. In the work on the mPCV-TWPA concept, published in Ref. [94], I proposed this technique as a general strategy to mitigate the issue of sidelobes dominating the reflections in TWPAs with periodic circuit parameter variation. It was further investigated with the help of the cascaded transfer matrices method and WRspice simulations by **Antoine Andrieux** during his internship in our working group.

Sidelobes can be understood as the frequency-domain response to the abrupt onset of periodic circuit parameter variation in the space domain, as illustrated in Fig. 3.17a. Considering an impedance $Z_0 = 50 \Omega$ before and after the TWPA, the uniform capacitance variation of the TWPA (constant ζ_1 and ζ_2) can be seen as having a rectangular envelope. The Fourier transform of the rectangular-pulse function in the space domain is a sinc function in the frequency domain, which is charac-

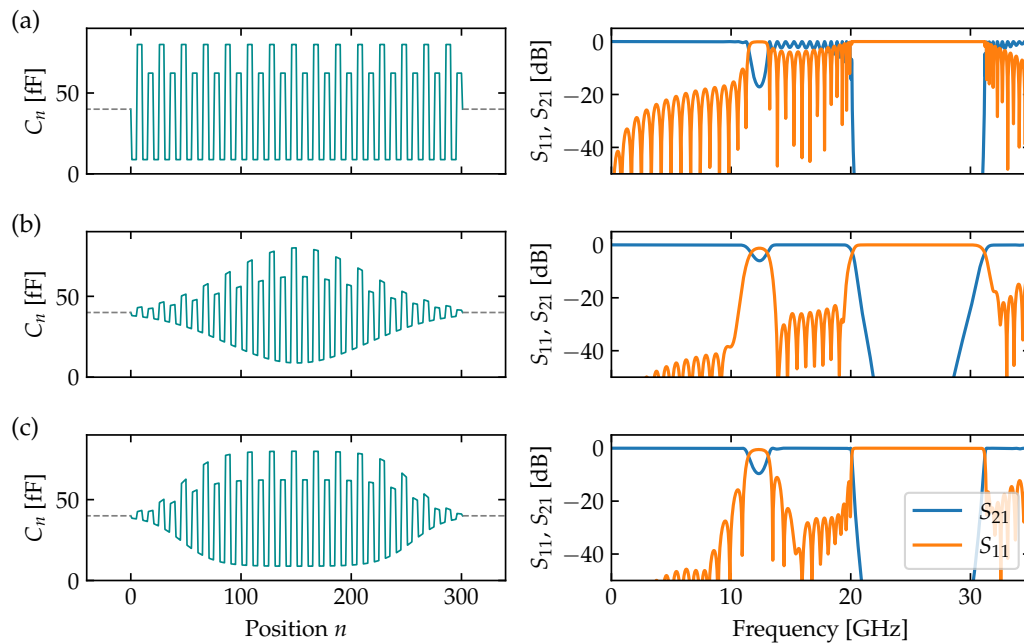


Fig. 3.17.: Illustration of the origin of sidelobes and their suppression via apodized capacitance variation. For better visibility short TWPA circuits with 300 unit cells are analyzed. Grey dashed lines denote the exterior circuitry with a reference impedance $Z_0 = 50 \, \Omega$. **(a)** Uniform multiperiodic capacitance variation. The rectangular envelope of the capacitance profile in the space domain leads to sinc-function-like sidelobes in the frequency domain. **(b)** More than 20 dB sidelobe suppression via enveloping the capacitance profile with a Gaussian function. **(c)** Compromise of both variants to increase the length of phase-matched parametric amplification. Scattering parameters are calculated using the cascaded transfer matrices method.

terized by one main lobe (corresponding to the stopband) and frequency-periodic sidebands. Enveloping the capacitance profile by another function allows to modify the frequency-domain response of the circuit. For example, the Fourier transform of a Gaussian function (Fig. 3.17b) is again a Gaussian function, which does not have any sidelobes. Since the Gaussian function is truncated, residual sidelobes appear. Nevertheless, by Gaussian apodization, sidelobes can be suppressed by orders of magnitude.

Apodization affects dispersion engineering inside the TWPA, because the modification of the dispersion relation qualitatively follows the apodization function. Considering mPCV, this mainly results in insufficient suppression of the second harmonic of the pump at the beginning and the end of the circuit due to small dispersion. It is observed in circuit simulations that the pump amplitude is therefore severely reduced already in the first few hundreds of unit cells. This obstacle can be solved in two ways. The first way is a combination of smooth exponential

enveloping and uniform capacitance variation (Fig. 3.17c). In this example, the envelope function is based on the Fermi function, and reads

$$a_F(n) = \frac{1}{1 + \exp\left(\frac{n_0 - n}{\sigma_F}\right)} + \frac{1}{1 + \exp\left(\frac{n - (N - n_0)}{\sigma_F}\right)} - 1, \quad (3.38)$$

where n_0 and σ_F are parameters determining the shape. It presents a compromise of a smooth transition from Z_0 to the periodically varied impedance, and a long section of uniform capacitance variation for dispersion engineering. Compared to Gaussian apodization, the residual sidelobes are somewhat more pronounced particularly close to the main stopband, but still much suppressed compared to uniform capacitance variation (compare Fig. 3.18 to Fig. 3.16).

A second way to reduce parasitic effects like second-harmonic generation is to additionally envelope the nonlinearity of the circuit. In an rf-SQUID-based TWPA this can be done via decreasing the critical currents of the Josephson junctions (grey line in Fig. 3.18) to reduce the parameter β_L , with only a small effect on the linear properties, since L_m is kept constant. This nonlinearity enveloping is similar to the Floquet-TWPA concept proposed by Peng et al. [98], a Josephson-junction-based 4WM TWPA where the critical currents of the Josephson junctions are varied with a truncated Gaussian. This variation corresponds to a similarly varied effective pump strength, with a remarkable consequence of decoupling the signal and idler waves from their sidebands. In our circuit, similar effects as described in Ref. [98] were observed, but not as pronounced because of a smaller level of nonlinearity variation. It should be noted, however, that the realization of a practical Floquet-mode TWPA could be somewhat easier using rf-SQUIDs instead of Josephson junctions, because the former do not limit the usable pump current as drastically, and because the parameter β_L presents an additional degree of freedom, effectively decoupling the linear and nonlinear characteristics of the circuit.

We combine both approaches via the capacitance and critical-current profiles as depicted in Fig. 3.18. The resulting gain profile, together with the reflection coefficient, is shown in Fig. 3.19. The stopband of the apodized TWPA is wider to achieve phase matching, so the gain profile is somewhat narrower. In contrast to the mPCV-TWPA of Sec. 3.2.4 (purple curves), the gain profile is virtually free of gain ripple. More importantly, the backaction unto the signal source, determined by the reflection coefficient, is reduced to values below or close to -15 dB, comparable to that of passive microwave components, whereas it approaches values as high as $+8$ dB (larger than unity) for the mPCV-TWPA with uniform capacitance variation, detrimental for sensitive signal sources.

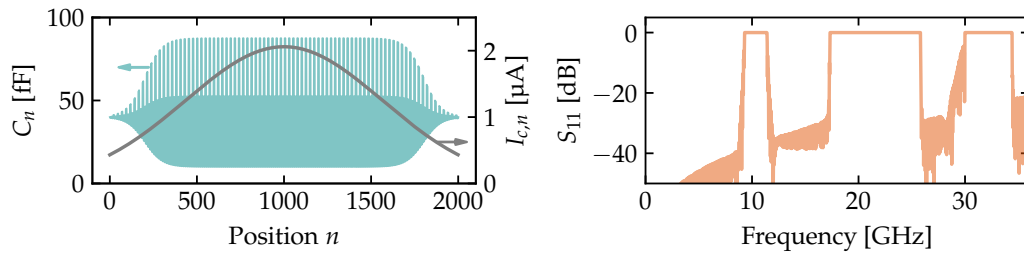


Fig. 3.18.: Capacitance profile of an apodized mPCV-TWPA and resulting small-signal reflection coefficient. Additionally, the critical currents of the rf-SQUIDs are varied according to the grey curve. The capacitance variation results in strong sidelobe suppression. Parameters are $\sigma_F = 50$, $n_0 = 200$, $\zeta_1 = 0.25$, $\zeta_2 = 0.75$, $m = 24$, $N = 2000$, $L_g = 100$ pH, and mean capacitance $C_0 = 40$ fF.

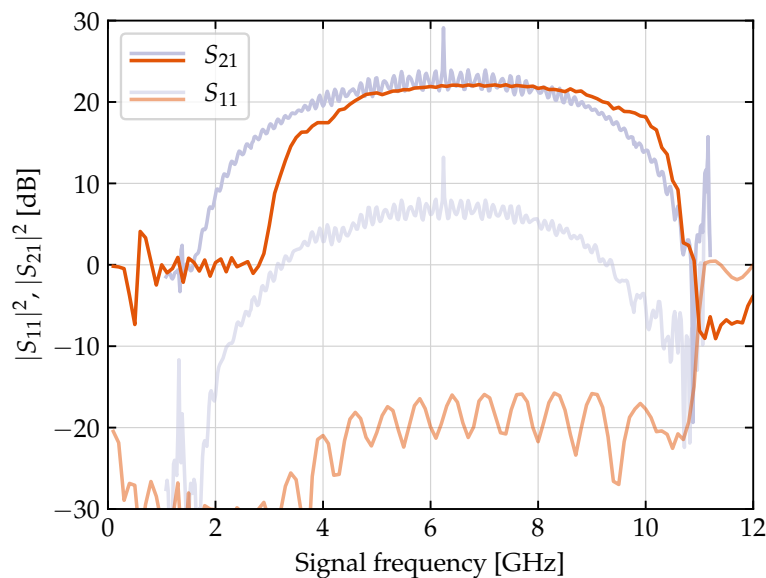


Fig. 3.19.: Power gain and reflection coefficient of mPCV-TWPAs with apodized (orange) and uniform (purple) capacitance variation. The C_n and $I_{c,n}$ profiles of the apodized circuit are shown in Fig. 3.18. The gain profile of the apodized TWPA is almost free of gain ripple. While the reflection coefficient of the conventional TWPA is larger than unity, that of the apodized TWPA is more than 20 dB smaller. WSpice simulation parameters of the mPCV-TWPA without apodization see caption of Fig. 3.12, but with $I_p = 1.7$ μ A. Simulation parameters of the apodized circuit are $I_p = 2$ μ A, $f_p = 13.7$ GHz, and $I_{dc} = 6.49$ μ A.

In this way, the effects of gain ripple, backaction, and instability, all enhanced by sidelobes, can be greatly reduced via apodization. Since sidelobes can be strongly suppressed, and the average impedance of the rf-SQUID-based TWPA can be tuned by magnetic flux to match it to its embedding circuitry, other factors may now limit impedance matching. These factors include frequency-dependent port impedances of external microwave components, bondwire inductances, or reflections inside a real TWPA device caused by unit-cell-to-unit-cell variations due to fabrication tolerances.

The apodization technique is not limited to 3WM TWPAs, but can be applied in the same way to 4WM TWPAs, and as well to kinetic-inductance-based TWPAs. Since periodic loading or periodic circuit parameter variation are among the most prominent phase-matching techniques for TWPAs, apodization may leverage TWPA development towards greatly reduced backaction and higher achievable gains.

3.4 Saturation power

One of the main figures of merit of an amplifier is its saturation power, or gain compression point, marking the upper bound of signal powers which can still be amplified linearly. Together with the noise level, the saturation power determines the dynamic range of an amplifier. In the context of frequency-multiplexed readout of large numbers of superconducting qubits [65] or detector arrays [129, 130] with the use of parametric amplifiers, two amplifier figures of merit are crucial: First, the bandwidth of the amplifier facilitating frequency-multiplexing of many channels, and second, its power-handling capability to simultaneously read out all channels with one readout tone each.

Due to the distributed nature of their traveling-wave architecture, TWPAs have enabled a tenfold increase of saturation power compared to JPAs with resonator-based architectures, as demonstrated in Ref. [49]. The same is observed for the bandwidth. However, it should be noted that resonant JPAs improved significantly over the last decade due to impedance engineering and their use of arrays of Josephson junctions or SQUIDs instead of single nonlinear elements, so their saturation powers approach those achieved by TWPAs [66, 67, 51].

Saturation in TWPAs is accompanied by noise rise and intermodulation distortion, leading to crosstalk and reduction of readout fidelity of frequency-multiplexed qubit readout [131]. In that work, a noise rise was observed to begin already 10 dB below the saturation power, the reason of which is a matter of ongoing research. To mitigate such deteriorating effects, it is recommendable to keep an overhead of, say, 10 dB to the saturation power of the amplifier. Then, considering a readout power of -120 dBm per channel (see e.g. Ref. [131]) and an overhead of 10 dB, a TWPA should have a saturation power of -100 dBm to read out 10 qubits simultaneously, or -90 dBm for 100 qubits. Therefore, the development of parametric amplifiers with high saturation powers is key for scaling up quantum computers.

3.4.1 Saturation mechanisms

Nonlinearly altered phase matching

Several effects can cause saturation in parametric amplifiers. In resonator-based JPAs the Kerr nonlinearity leads to a power dependent shift of the resonant mode. A sufficiently large signal amplitude therefore can shift the resonance away from the tones involved in parametric amplification. Since resonator-based JPAs typically are rather narrow-band, a small shift may be sufficient to drastically reduce gain. In this way, in resonant 4WM parametric amplifiers it is the Kerr nonlinearity that both enables amplification and mediates gain compression [132, 133]. Kerr-free 3WM

parametric amplifiers circumvent this issue and achieve tenfold larger saturation powers than their 4WM counterparts [134].

In the case of TWPAs, the Kerr nonlinearity causes a nonlinear phase shift due to SPM and XPM [56], and hence a phase mismatch for parametric amplification, usually compensated by dispersion engineering techniques to achieve large gain. Since this phase mismatch is nonlinear, it can be altered by sufficiently large signal powers. When this occurs, phase matching conditions can be better or worse than in the low-signal-power case, leading to increased or decreased gain. In both cases, at sufficiently large signal amplitudes the signal gain is no longer constant, marking the upper edge of linear amplification.

In a similar manner, the 3WM effect of rectification of waves with high amplitudes by noncentrosymmetric nonlinearity (refer to Fig. 2.5) may shift the phase-matching conditions and lead to signal-power-dependent gain. Note that the rf-SQUID-based TWPA presented in this thesis possesses flux-bias points where rectification and Kerr effects cancel each other out, such that the power dependence of the rf-SQUID inductance, and hence of phase matching, is small (experimentally investigated in Fig. 4.10). Moreover, the range of pump frequencies where phase matching is approximately fulfilled is rather wide (c.f. Fig. 4.16), so a small signal-power-induced shift may not drastically alter the gain. These are factors why high saturation powers are achieved with that device. An effect of potentially much larger impact on gain saturation in TWPAs is pump depletion [56].

Pump depletion

Ideally, the signal wave in a TWPA increases exponentially in amplitude while propagating along with the pump wave (solid lines in Fig. 3.20). When the input signal power, or the device length, is large, the signal amplitude approaches that of the pump, and the amplitude reduction of the pump wave due to power transfer to signal and idler is no longer negligible (dashed lines). The signal starts depleting the pump, which can no longer sustain exponential signal growth, and the gain drops.

For larger signal powers, nearly the full pump power can be transferred to signal, idler, and other mixing products, before power transfer reverses and these tones feed the pump (dotted lines in Fig. 3.20). Not only is this accompanied by drastically reduced signal gain, but also by the presence of signal and idler harmonics and a multitude of their mixing products, causing harmonic and intermodulation distortion [85]. Therefore, signal powers close to or above the 1-dB compression point, where the gain reduced by 1 dB relative to the unsaturated gain, are not a favorable regime for most applications.

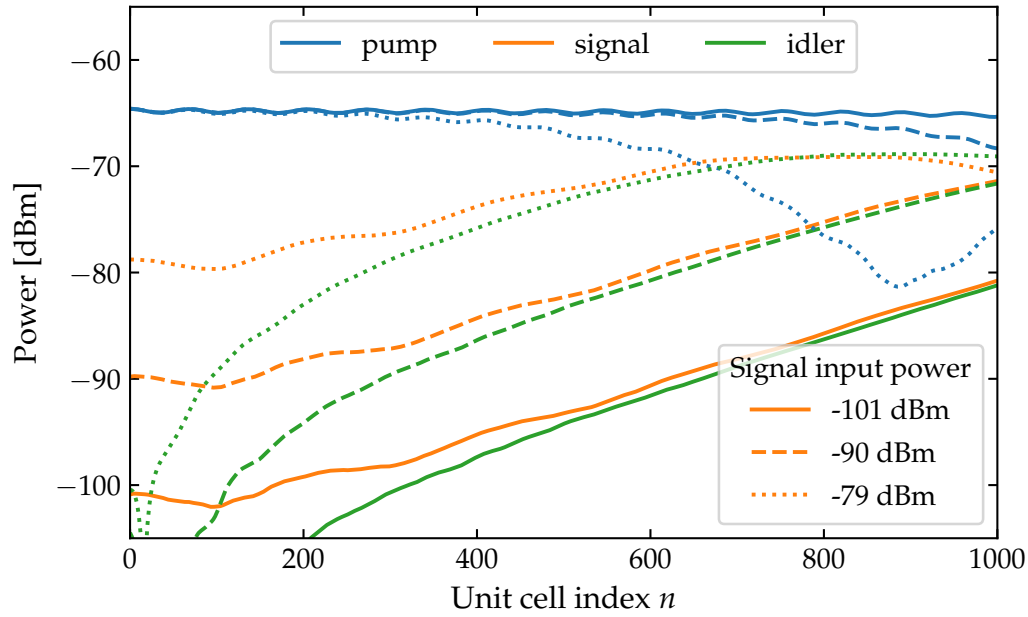


Fig. 3.20.: Gain saturation by increased signal power. Shown are simulations of the spatial evolution of the powers of pump, signal, and idler for increased signal input power. The dashed lines roughly correspond to a signal power where the gain is reduced by 1.6 dB. For larger signal powers nearly complete power transfer from pump to signal, idler, and other mixing products, is possible (dotted line), accompanied by drastically reduced gain. Undulations on signal and pump stem from interactions with up-conversion products and second harmonic. The investigated circuit is an RPM-TWPA with $\beta_L = 0.3$, with corresponding signal power dependences and simulation parameters given in Fig. 3.21.

3.4.2 Analytical approximation of gain compression

Manley-Rowe relations

In a 3WM TWPA, the evolution of wave amplitudes are governed by the Manley-Rowe relations [42], representing energy conversion in ideal parametric processes. Considering ideal three-mode parametric amplification with $\omega_p = \omega_s + \omega_i$, the corresponding wave powers at any position in the TWPA are [135]

$$\frac{P_s - P_s^{\text{in}}}{\omega_s} = \frac{P_i}{\omega_i} = \frac{P_p^{\text{in}} - P_p}{\omega_p}, \quad (3.39)$$

or equivalently, in terms of corresponding numbers of photons,

$$N_s - N_s^{\text{in}} = N_i = N_p^{\text{in}} - N_p, \quad (3.40)$$

where index ‘in’ denotes the respective input power. In the case of complete conversion of the pump power to signal and idler, i.e. full pump depletion, $P_p = 0$ at the output, maximum signal output power is achieved,

$$P_s^{\max} = P_s^{\text{in}} + \frac{\omega_s}{\omega_p} P_p^{\text{in}} \quad (3.41)$$

It is marked by a vertical dotted line in Fig. 3.21, presenting signal and pump transmission corresponding to the simulations shown in Fig. 3.20 for a range of signal powers. The resulting signal gain at complete pump conversion is $G = 1 + (P_p^{\text{in}}/\omega_p)/(P_s^{\text{in}}/\omega_s)$.

Gain compression by partial pump depletion

Describing partial pump depletion requires a more involved approach, solving the coupled-mode equations, Eqs. 2.78–2.80, which was done by Zorin for the rf-SQUID-based 3WM TWPA in the absence of phase mismatch [57]. The resulting signal power gain, depending on the signal input power, is

$$G = \frac{1}{\text{dn}^2(gN, k)} \xrightarrow{P_s^{\text{in}} \ll P_p^{\text{in}}} \cosh^2 gN, \quad (3.42)$$

with the Jacobi elliptic function $\text{dn}(u, k)$ (see e.g. Ref. [136]) and the modulus (using an upright symbol to distinguish modulus k from wavenumber k)

$$k = \frac{1}{\sqrt{1 + \omega_p P_s^{\text{in}}/\omega_s P_p^{\text{in}}}} \xrightarrow{P_s^{\text{in}} \ll P_p^{\text{in}}} 1. \quad (3.43)$$

To obtain a more illustrative expression for the saturated gain, I propose the following approximation. Considering high undepleted gain $G_0 = \cosh^2 gN \gg 1$ (Eq. 2.94) on the order of 20 dB, with $u = gN \approx 3$, signal input powers are two to three orders of magnitude lower than pump powers even at complete pump conversion (Eq. 3.41). This is because, as it turns out below, the 1-dB compression power is $P_s^{1\text{dB}} \approx P_p/4G_0$. Therefore, the modulus is deep in the limit $k \approx 1$. Under these conditions, the approximations of Ref. [136] for Jacobi elliptic functions can be further simplified,

$$\begin{aligned} \text{dn}(u, k) &\approx \frac{1}{\cosh(u)} + \frac{k'^2 \tanh(u)}{4 \cosh(u)} [\sinh(u) \cosh(u) + u] \\ &\approx \frac{1}{\cosh(u)} \left[1 + \frac{k'^2}{4} \cosh^2(u) \right]. \end{aligned} \quad (3.44)$$

The complement modulus of k is defined as

$$k' = \sqrt{1 - k^2} = \frac{1}{\sqrt{1 + \omega_s P_p^{\text{in}} / \omega_p P_s^{\text{in}}}} \xrightarrow{P_s^{\text{in}} \ll P_p^{\text{in}}} \sqrt{\frac{\omega_p P_s^{\text{in}}}{\omega_s P_p^{\text{in}}}}. \quad (3.45)$$

By plugging in the undepleted gain $G_0 = \cosh^2 gN$ into Eq. 3.44, we obtain a simple approximation of gain saturation due to pump depletion,

$$G = \frac{G_0}{1 + G_0 f_p P_s^{\text{in}} / 2 f_s P_p^{\text{in}} + (G_0 f_p P_s^{\text{in}} / 4 f_s P_p^{\text{in}})^2} \quad (3.46)$$

$$\approx \frac{G_0}{1 + G_0 f_p P_s^{\text{in}} / 2 f_s P_p^{\text{in}}} \quad (3.47)$$

As presented in Fig. 3.21, Eq. 3.46 is in very good agreement with the full expression, Eq. 3.42, even for several dB of gain compression. While the further simplified approximation, Eq. 3.47, deviates significantly at high pump depletion, it gives good results for gain saturation of up to 1–2 dB, the most relevant region for practical considerations, because it allows estimating the 1-dB compression point.

To examine the validity of the used three-mode model in the context of gain saturation and pump depletion, Fig. 3.21 also shows a WRspice simulation of the 3WM TWPA, with internal dynamics presented above (Fig. 3.20). Due to biasing for Kerr-free conditions, the absence of SPM and XPM is expected. Agreement within ~ 1 –2 dB is found between analytical model and simulation. While the accompanying pump depletion presents the dominant saturation mechanism, these deviations indicate the presence of additional mechanisms, such as higher-order mixing products, or nonlinear phase mismatches stemming from rectification due to noncentrosymmetric nonlinearity, described in Sec. 3.4.1, which also causes residual SPM and XPM.

The decreased pump transmission (Fig. 3.21b) can be calculated using the Manley-Rowe relations, Eq. 3.39, via substituting $P_s^{\text{out}} = G P_s^{\text{in}}$ and plugging in the gain approximation Eq. 3.46,

$$S_{21,p}^2 \approx \frac{P_p^{\text{out}}}{P_p^{\text{in}}} = 1 + \frac{\omega_p}{\omega_s P_p^{\text{in}}} (1 - G) P_s^{\text{in}}. \quad (3.48)$$

The good agreement even beyond the point of complete pump conversion (dotted grey line, Eq. 3.41) again exemplifies the accuracy of Eq. 3.46.

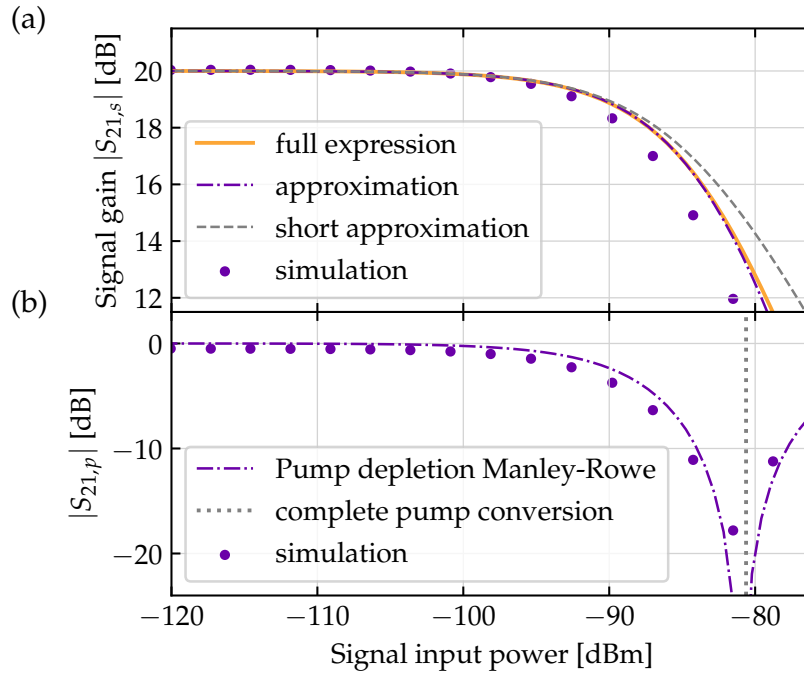


Fig. 3.21.: Dependence of signal gain $|S_{21,s}|$ and pump transmission $|S_{21,p}|$ on the signal input power $|a_{1,s}|^2$. **(a)** Dots show the simulation of an RPM-TWPA circuit. Lines depict analytical expressions for gain saturation, Eq. 3.42 (solid), Eq. 3.46 (dash-dotted), and Eq. 3.47 (dashed). Simulations saturate at slightly lower powers than theory predicts, suggesting additional saturation mechanisms apart from pump depletion. **(b)** Gain saturation is accompanied by decreased pump transmission (c.f. Fig. 3.20 for the internal dynamics). The dash-dotted line is calculated by Eq. 3.48 using the same approximation as in Panel (a). Circuit parameters see Tab. 3.2, but with $L_m = 70$ pH, $\beta_L = 0.3$, and $I_c = \varphi_0 \beta_L / L_m$, and for a signal frequency of 7 GHz, a pump frequency of 13.3 GHz, and $I_{dc} = 8.8$ μ A.

1 dB compression point in three-wave mixing and four-wave mixing TWPAs

As a convention, the signal power for which the gain is reduced by 1 dB relative to the unsaturated gain, is used to specify the saturation power, called the 1-dB compression point [85]. It can be arbitrarily referred to the input or the output of the amplifier. For many semiconductor-based amplifiers the compression point is referred to the output, useful to estimate which maximum output power the amplifier is capable to emit to a load, for example, to an antenna. On the other hand, for superconducting parametric amplifiers the compression point is typically referred to the input, useful to estimate which maximum input power coming from a signal source, e.g. qubits or detectors, can be linearly amplified.

With Equation 3.47 we have obtained an expression which we can directly compare to its 4WM counterpart. Considering $\omega_p \approx 2\omega_s$ (center of the gain band), it reads

$$G = \frac{G_0}{1 + G_0 P_s^{\text{in}} / P_p}. \quad (3.49)$$

By rearranging, the 1-dB compression point, $G = 10^{-1/10} G_0$, can be found as

$$P_{s,\text{sat}}^{\text{dBm}} \approx \frac{P_p}{4G_0} = P_p^{\text{dBm}} - G_0^{\text{dB}} - 6 \text{ dB}, \quad (3.50)$$

presenting a simple relation between pump power, power gain, and saturation power.

An expression similar to Eq. 3.49 was derived by Kylemark et al. [137] for a 4WM optical-fiber TWPA,

$$G = \frac{G_0}{1 + 2G_0 P_s^{\text{in}} / P_p}, \quad (3.51)$$

where $\omega_s \approx \omega_p$, G_0 is the unsaturated power gain for small signals, and P_p denotes the undepleted (input) pump power. This equation covers the effect of high signal power and pump depletion on SPM and XPM. Although derived for full pump depletion, it still gives good approximations also for partial power transfer [59]. The resulting 1-dB compression point,

$$P_{s,\text{sat}}^{\text{dBm}} \approx \frac{P_p}{8G_0} = P_p^{\text{dBm}} - G_0^{\text{dB}} - 9 \text{ dB}, \quad (3.52)$$

is 3 dB lower than that of a 3WM TWPA. This difference originates in impaired phase match by SPM and XPM for increasing signal power, additional to pump depletion. Note that in 4WM, the Kerr nonlinearity both facilitates parametric amplification and causes SPM and XPM, in contrast to 3WM, where the latter effects are small and can often be neglected.

Equations 3.50 and 3.52 allow a simple estimation of the ideal saturation power of a TWPA. Simulations of various 3WM RPM-TWPA and mPCV-TWPA circuits with different circuit parameters, and three experimentally implemented TWPAs, 3WM and 4WM, all having gains close to 20 dB, are compiled in Fig. 3.22 (refer to Fig. 4.30 for more experimental implementations). The 3WM simulations show agreement within ~ 2 dB with Eq. 3.50. Deviations can be attributed to $f_s \neq f_p/2$ (with $f_s = 7 \text{ GHz} > f_p/2$ in most simulations), gains slightly deviating from 20 dB, residual XPM and SPM in the Kerr-free circuits, the presence of loss in the experimental devices, and the presence of mixing products beyond the three-mode model.

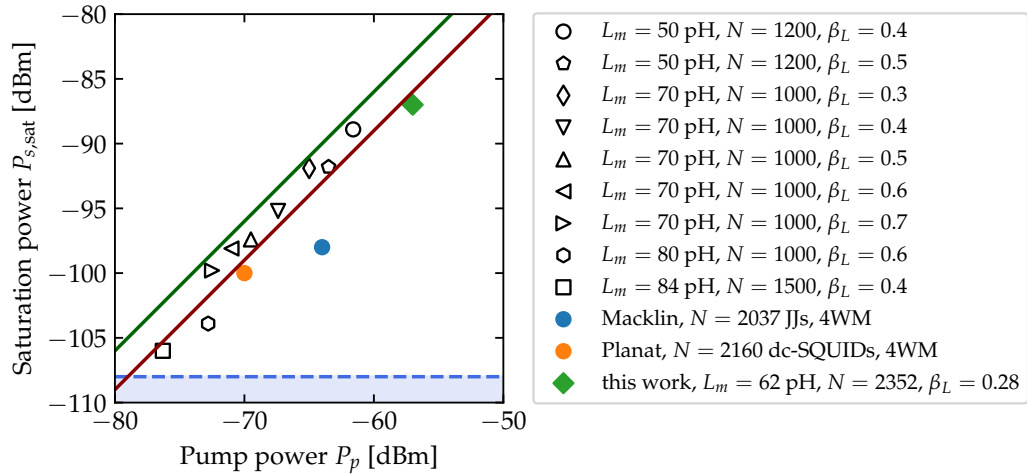


Fig. 3.22.: Pump powers and saturation powers of various TWPA circuits with gains close to 20 dB. Higher pump power enables higher saturation power, roughly following Eq. 3.50 (solid green line) for 3WM and Eq. 3.52 for 4WM. Open black symbols denote simulations of 3WM RPM-TWPAs with different circuit parameters, except the square symbol denoting the mPCV-TWPA of Sec. 3.2.4. For comparison, filled colored symbols represent experimental 4WM devices of Refs. [49, 50] and the rf-SQUID-based 3WM TWPA of Chapter 4. A lower limit on the required saturation power is set by the power of vacuum fluctuations in the TWPA bandwidth, shown for a bandwidth of 4 GHz (blue dashed line).

3.4.3 Saturation by quantum and thermal noise

Another fundamental constraint is placed by quantum noise, or vacuum fluctuations, which are incident on the amplifier even in the absence of any signal [125]. At each frequency, the noise amplified by the amplifier is at least $\hbar\omega$ [5], as further detailed in Sec. 4.6. Integrating this power spectral density over the entire bandwidth B of the amplifier results in a power $P_{\text{noise}} \approx hf_p B/2$ for a 3WM amplifier, where the gain band is centered around $f_p/2$. For a bandwidth of 4 GHz and $f_p = 12$ GHz, $P_{\text{noise}} = -108$ dBm, depicted as horizontal line in Fig. 3.22. For thermal noise relevant for operation at higher temperatures, $k_B T \gg \hbar\omega$, this fundamental limit is even higher, $P_{\text{noise}} \approx k_B T B$, which is -96 dBm at 4.2 K for a bandwidth of 4 GHz.

For the mPCV-TWPA, discussed in Sec. 3.2.4, $P_{\text{vac}} = -106$ dBm due to its exceptionally large bandwidth of ~ 6 GHz. Unfortunately, the simulated saturation power of that circuit, depicted as a square symbol, is also -106 dBm. Hence, in a real experiment, vacuum fluctuations would suffice to drive this circuit into saturation. To prevent such shortcomings, the circuit parameters were adjusted to enable a well-performing experimental implementation of the mPCV-TWPA (solid green diamond). For this purpose, a simple analytical model is presented in the following, enabling the design of a TWPA for a targeted saturation power.

3.4.4 Designing an rf-SQUID-based TWPA with targeted saturation power

To design an rf-SQUID-based 3WM TWPA with a targeted saturation power, we consider the power gain around the center of the gain profile, $f_s \approx f_p/2$ (but ignoring phase-sensitive effects at $f_s = f_p/2$), Eq. 2.106,

$$G_0 = \cosh^2 g_0 N, \quad (3.53)$$

assuming that any existing phase mismatch is compensated by dispersion engineering techniques. To achieve a targeted gain of 20 dB, or $G = 100$, the required exponent is $g_0 N = \text{arcosh} \sqrt{G_0}$, with the exponential gain coefficient given by Eq. 2.96,

$$g_0 = \frac{1}{4} |\beta| |A_{p0}| k_p^2. \quad (3.54)$$

In this equation we can approximate $k_p \approx \omega_p L / Z$, with $Z = \sqrt{L/C_0} = Z_0 = 50 \Omega$ for impedance-matching to the circuitry embedding the TWPA. The pump frequency $f_p = \omega_p / 2\pi$ determines the position of the gain band, centered around $f_p/2$. Further, we chose a flux bias such that $\phi_{dc} = \pi/2$, tuning the TWPA to its Kerr-free regime. At this bias, the rf-SQUID inductance is $L = L_m$ and the quadratic-nonlinearity coefficient is $\beta = \beta_L/2$ in the case of ideal rf-SQUIDs.

Recall that the pump amplitude $|A_{p0}|$ represents an oscillation amplitude of the superconducting phase on the input node of the circuit, $n = 0$. To link it to pump current and pump power, we first have to convert the node-phase amplitude to the amplitude of the phase drop across a unit cell, $\phi(t) = \phi_{n+1}(t) - \phi_n(t)$ (for ideal rf-SQUIDs equivalent to the phase drop across the Josephson junction of the SQUID). The phase difference of a wave across two adjacent nodes is equal to the dimensionless wavenumber k , see Sec. 2.2.2. Assuming harmonic time dependence and spatially slowly varying amplitudes, such that two neighboring nodes have approximately the same amplitude, $A_n \approx A_{n+1}$, the phase drop ϕ across one unit cell n is

$$\begin{aligned} \phi(t) &= \phi_{n+1}(t) - \phi_n(t) \\ &= A_n [e^{j(k(n+1)-\omega t)} - e^{j(kn-\omega t)}] \\ &= A_n e^{j(kn-\omega t)} [e^{jk} - 1] \\ &\approx jk A_n e^{j(kn-\omega t)} \end{aligned} \quad (3.55)$$

for $k \ll 1$. Therefore, the amplitude of the phase drop $\phi(t)$ is $k|A_n|$ (note that node-phase amplitudes are complex in the notation of the coupled-mode theory). In

this way, we can link $|A_{p0}|$ to ϕ_{p0} , and via the small-signal inductance to the pump current,

$$\phi_{p0} = k_p |A_{p0}| = LI_p / \varphi_0. \quad (3.56)$$

Although there is no maximum pump current, above which the rf-SQUID turns to the voltage state, there indeed is a maximum useful pump current, above which the TWPA behaves unpredictably. It roughly corresponds to a maximum useful phase drop of $\phi_{p0} \sim 1$ rad [57], c.f. Fig. 2.5. For $\beta_L < 0.5$ this is in empirical agreement with simulations of various TWPA circuits having different parameters, while for $\beta_L > 0.5$ higher-order effects and unpredictable behavior emerges at somewhat lower amplitudes (c.f. Fig. 2.4).

The current flowing through the rf-SQUID I_{SQ} relates to the input current I_{in} of a unit cell, seen as a two-port, via

$$\frac{I_{SQ}}{I_{in}} = 1 - \frac{\omega^2}{\omega_0^2} + \frac{j\omega/\omega_0}{1 + j\omega/\omega_0}. \quad (3.57)$$

For small frequencies, the current flowing through the SQUID can be equated to the input (and output) current of the two-port, which is valid within $\pm 5\%$ in $\omega < 0.7\omega_0$. Then, considering the pump current we can write $I_{in} \approx I_{SQ} = I_p$ and find the pump power

$$P_p = \frac{1}{2} I_p^2 Z. \quad (3.58)$$

Finally, using the approximate relation between pump power and saturation power, Eq. 3.50, we can express the saturation power via the pump amplitude ϕ_{p0} ,

$$P_{s,sat} = \frac{P_p}{4G_0} = \frac{I_p^2 Z_0}{8G_0} = \frac{Z_0}{8G_0} \left(\frac{\varphi_0 \phi_{p0}}{L_m} \right)^2, \quad (3.59)$$

relating geometrical inductance L_m to the saturation power for a given value of ϕ_{p0} (Fig. 3.23). In fact, it turns out that L_m is the circuit parameter with the strongest impact on gain and saturation power, because it determines both linear and nonlinear properties, such as phase velocity and impedance on the one hand, and nonlinearity coefficients on the other. Other circuit parameters, e.g. ground capacitances C_0 or critical current I_c , influence in a first-order approximation either only linear or only nonlinear properties. This has strong implications on the practical realization of rf-SQUID-based TWPAs, as discussed in Sec. 3.5.3. The latter two quantities are determined by the choice of L_m via $C_0 = L_m / Z_0^2$ and $I_c = \varphi_0 \beta_L / L_m$.

The remaining degrees of freedom are the screening parameter β_L , determining the critical current I_c , and the length of the TWPA circuit, i.e., the number of unit

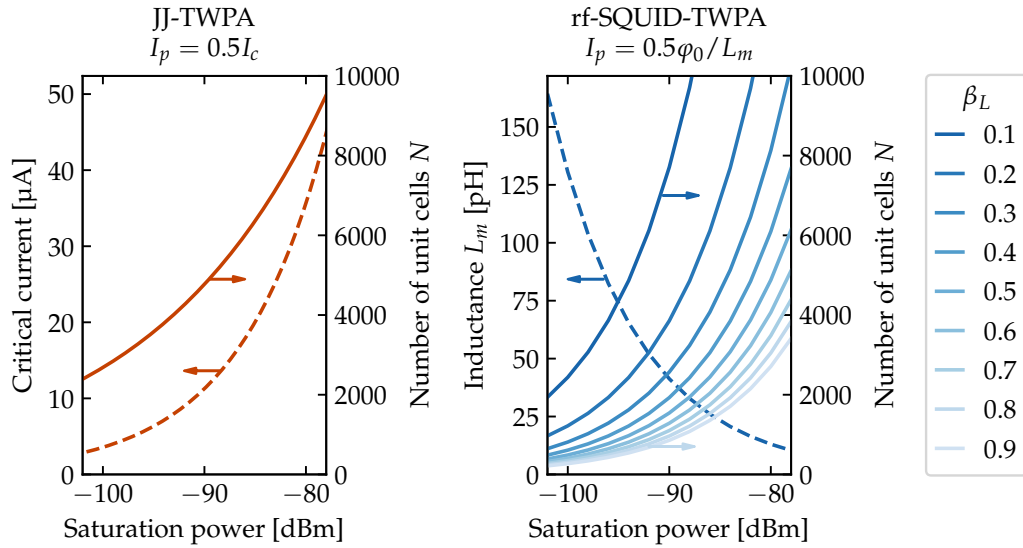


Fig. 3.23.: Comparison of Josephson junction (JJ) based and rf-SQUID based TWPA regarding saturation power. Pump amplitudes are fixed to $I_p = 0.5I_c$ (JJ-TWPA) and $I_p = \phi_0\phi_p/L_m$, with $\phi_p = 0.5$ rad (rf-SQUID-TWPA), respectively, relating the critical current and the geometrical inductance to the saturation power (dashed lines). Solid lines show the necessary number of unit cells N to achieve a gain of 20 dB. For rf-SQUID based TWPA the parameter β_L represents a degree of freedom with strong impact on the required N . Note that large deviations from these curves are possible for nonideal rf-SQUIDs, see main text. Parameters are $f_p = 6$ GHz (4WM) and $f_p = 12$ GHz (3WM).

cells N . From Eqs. 3.53 and 3.54, we obtain an expression for the values of β_L and N required to achieve the targeted gain G_0 ,

$$\beta_L N = \frac{8Z_0 \text{arccosh} \sqrt{G_0}}{\phi_{p0} \omega_p L_m}. \quad (3.60)$$

With a choice of β_L and N (solid lines in Fig. 3.23), all parameters of the circuit are determined. For proper dispersion engineering, this workflow was found to lead to good agreement with WRspice simulations within usually less than 3 dB for the targeted parameters G_0 and $P_{s,\text{sat}}$.

Nonideal rf-SQUIDs

If nonideal rf-SQUIDs are considered instead of ideal rf-SQUIDs, their theory presented in Sec. 2.1.3 and Appendix A.1, a few modifications are necessary. The SQUID inductance is now given by Eq. 2.38. Crucially, the presence of (linear) feedline inductance L_f increases the inductance of one unit cell, and decreases the participation ratio, i.e. the ratio of (nonlinear) SQUID inductance and total cell inductance. As a consequence, the amplitude of the phase drop ϕ_{p0} across

the Josephson junction is no longer identical to that across the unit cell, $\hat{\phi}_{p0}$, but follows Eq. A.32. In the case of biasing at $\phi_{dc} = \pi/2$ (note that this is not exactly the Kerr-free point for nonideal rf-SQUIDs, see Eq. 2.41), resulting in the cell inductance $L = L_f + L_m$ and the participation ratio $p = L_m / (L_f + L_m) < 1$. Because of the reduced participation ratio, the effective quadratic-nonlinearity coefficient is $\beta = p(1 - \alpha_f)\beta_L/2$ (Eq. 2.40), instead of $\beta_L/2$ for ideal rf-SQUIDs. Hence, the required pump amplitude for a targeted gain G_0 is larger,

$$\hat{\phi}_{p0} = \frac{8Z_0 \text{arcosh} \sqrt{G_0}}{p(1 - \alpha_f)\beta_L N \omega_p (L_m + L_f)} \quad (3.61)$$

resulting in a larger saturation power

$$P_{s,\text{sat}} = \frac{Z_0}{8G_0} \left(\frac{\varphi_0 \hat{\phi}_{p0}}{L_m + L_f} \right)^2. \quad (3.62)$$

This prediction is in rough agreement with experimental results (Sec. 4.5.4), with discrepancies due to the potentially large insertion losses at the pump frequency, not accounted for in the model.

Comparison of 3WM and 4WM regarding saturation power

Equation 3.60 is depicted in Fig. 3.23, along with the corresponding relations for a Josephson-junction-based 4WM TWPA. For the latter relations, Eq. 3.52 was used, together with Eqs. 12–14 and 23 from Ref. [59], to calculate the gain. In TWPAs based on Josephson junctions as nonlinear inductive elements, the parameters I_c , N and $P_{s,\text{sat}}$ are related in a unique way for a given maximum pump current, e.g. $I_p = 0.5I_c$. This is because linear and nonlinear properties are interrelated, since Josephson inductance and Kerr nonlinearity coefficient are both determined by I_c only (Sec. 2.1.1). In contrast, rf-SQUIDs offer an additional degree of freedom in circuit design via the parameter β_L , which effectively decouples linear and nonlinear properties. Apart from its impact on saturation power, the choice of the parameter β_L has further practical implications for the experimental realization of an rf-SQUID-based TWPA, as discussed in the next section.

3.5 Parameter spread in TWPAs

Any practical TWPAs implementation, consisting of hundreds to thousands of circuit elements, must exhibit sufficient robustness against the anticipated parameter variations of conventional fabrication procedures. The simulation study presented in the following was initially motivated by early works on RPM-TWPAs at PTB, the samples of which exhibited resonance-frequency spreads of several percent. Since PCV-TWPAs were investigated in our group simultaneously, an outstanding question towards the practical implementation of a dispersion-engineered 3WM TWPAs arose: Which dispersion-engineering candidate is better suited for reliable experimental implementation regarding the anticipated parameter spread of typical fabrication processes? A simulation study answered this question and paved the way towards the experimental demonstration of the TWPAs presented in Chapter 4.

TWPAs consist of vast numbers of ideally identical (or periodically varied) circuit elements, such as capacitors, Josephson junctions, or inductors. The deviation of this ideal case is parameter spread, precisely the random variation of circuit parameters around their targeted values. Parameter spread affects the performance of TWPAs in various aspects, including phase matching, impedance-matching, nonlinearities, etc. Earlier studies examined the impact of degraded phase matching resulting from resonance-frequency spread in RPM-TWPAs, either analytically via stochastic modeling [59] or employing Monte Carlo simulations on a linearized model [138]. The acceptable tolerance for critical-current spread of Josephson junctions was investigated through analytical techniques [139, 140] and harmonic-balance simulations [98].

In the work presented here, published in part in Ref. [95], two dispersion-engineering concepts, RPM and PCV, are rigorously analyzed and compared regarding their robustness against parameter spread, using circuit simulations. Additionally, research by Ó Peatáin et al. [141] independently and simultaneously utilized circuit simulations to investigate parameter spread and point defects in an rf-SQUID-based 3WM TWPAs without dispersion engineering, providing complementary insights to our work. While many of the aforementioned studies focused solely on individual aspects affected by parameter spread, our approach inherently encompasses the impact of spread on various factors such as phase matching, pump depletion, deviations from the optimal bias point, microwave reflections, etc., providing a more comprehensive and realistic analysis.

The following simulation study is based on the RPM-TWPAs and the PCV-TWPAs introduced in Sections 3.2.2 and 3.2.3, which were designed to be comparable in gain, bandwidth, and most of their circuit parameters (Tab. 3.2).

3.5.1 Vulnerability to resonance-frequency spread

To this point, the circuit parameters of all unit cells of a TWPA circuit model are identical, with the exception of C_n , exactly following the periodical capacitance profile, Eq. 3.21, of course. To investigate the effect of parameter spread on the amplification performance of the two TWPA concepts, we modify the circuit models by randomly varying the circuit parameters from unit cell to unit cell. Each of the parameters I_c, L_m, f_r , and the capacitances C_n, C_J, C_c , is varied isolatedly, keeping all other parameters at their zero-spread target values. Capacitances are varied simultaneously to emulate capacitance spread. The parameter spread follows a normal distribution, assuming that the parameters of all unit cells are uncorrelated.

Resonance-frequency spread

First, the effect of resonance-frequency spread of the $N/m = 250$ resonators in the RPM-TWPA is investigated (circuit parameters see Tab. 3.2). Neglecting for simplicity the influence of C_c (c.f. Eq. 3.16), the resonance frequency is given by the two parameters C_r and L_r as $f_r \approx 1/2\pi\sqrt{L_r C_r}$. Thus, the resonance-frequency spread can be emulated by varying the C_r values of the circuit with a normal distribution, having mean value $\langle C_r \rangle$ and standard deviation σ_{C_r} . The inductances L_r are kept constant, $\sigma_{L_r} = 0$. Then, the resulting normal distribution of the resonance frequencies exhibits a normalized standard deviation [142]

$$\frac{\sigma_{f_r}}{\langle f_r \rangle} = \frac{1}{2} \sqrt{\frac{\sigma_{L_r}^2 \sigma_{C_r}^2}{\langle L_r \rangle^2 \langle C_r \rangle^2} + \frac{\sigma_{L_r}^2}{\langle L_r \rangle^2} + \frac{\sigma_{C_r}^2}{\langle C_r \rangle^2}} = \frac{1}{2} \frac{\sigma_{C_r}}{\langle C_r \rangle}. \quad (3.63)$$

For every σ_{C_r} value, denoted as a sample, simulations are performed over a range of pump frequencies f_p . This is necessary because the pump frequency where phase matching is achieved, varies from sample to sample (c.f. Fig. 3.25c,d). For this range of pump frequencies, the highest achieved gain is extracted for each sample, shown in Fig. 3.24. A moving average with a window size of 20 samples is calculated to visualize the trend of the reduction of the attainable gain with increasing resonance-frequency spread.

The tolerable resonance-frequency spread can be expressed by the value where the gain reduces by 1 dB relative to the zero-spread case. The spread tolerance defined in this way is depicted in the inset of Fig. 3.24. The RPM-TWPA is more spread-tolerant for increasing coupling strength, set by the coupling capacitor C_c . Therefore, this degree of freedom in the design of the circuit can be utilized to render the TWPA more robust to parameter spread.

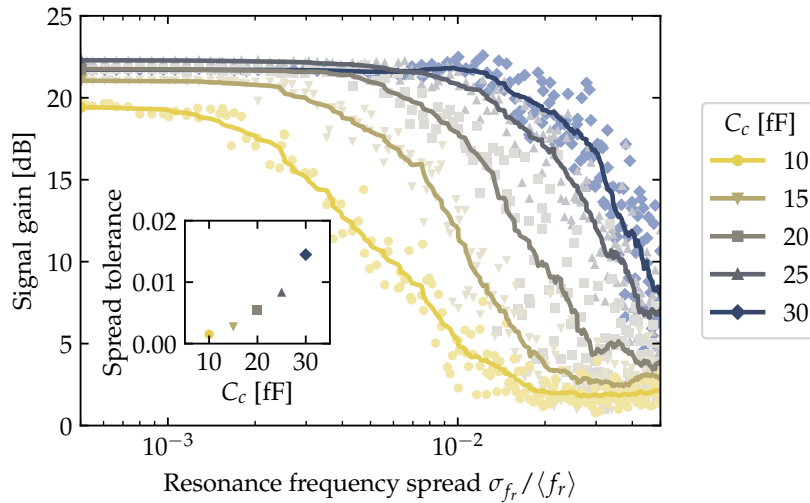


Fig. 3.24.: Effect of resonance-frequency spread on the signal gain of an RPM-TWPA. The resonance frequencies of each of the 250 individual resonators are sampled from a normal distribution with standard deviation σ_{f_r} and mean value $\langle f_r \rangle$. For each σ_{f_r} , the signal gain is simulated for a range of pump frequencies, 12.98–14.0 GHz, and the maximum achieved gain is plotted. For increasing σ_{f_r} , the achievable gain decreases, because phase matching of parametric amplification is hampered, as explained in Fig. 3.25. Solid lines are moving averages of 20 samples for illustration. The values of $\sigma_{f_r} / \langle f_r \rangle$ where the moving-average gain decreased by 1 dB relative to the zero-spread gain are plotted in the inset against C_c , which sets the coupling strength of the resonators. Stronger coupling leads to higher tolerance to resonance-frequency spread. Simulations were performed with an identical pump current of $I_p = 6.59 \mu\text{A}$ for comparability, and with a signal frequency of 7.7 GHz.

Small-signal simulations shed light on the mechanisms behind this behavior (Fig. 3.25). Shown is the small-signal transmission and wavenumber, both with and without resonance-frequency spread. Values of 2% capacitance spread and 1% inductance spread are used for the resonator capacitors and inductors, corresponding to 1.1% of resonance-frequency spread using Eq. 3.63, and representing realistic values for lumped element resonators [125, 143, 144, 145]. Resonance-frequency spread reduces the dispersive modification of the wavenumber while widening the transmission dip. Whether phase matching without severe pump depletion can be achieved is determined by the coupling strength of the resonators to the transmission line. Alternatively, increasing the number of resonators by decreasing their period m has the same effect of making the TWPA more tolerant to resonance-frequency spread. For example, spread tolerances of $> 2\%$ were reported for $m = 1$ (one resonator per unit cell) in Refs. [59, 138]. However, this comes at the cost of increased footprint on the chip. On the other hand, the resonance-frequency spread can be reduced by using distributed resonators, instead of lumped-element

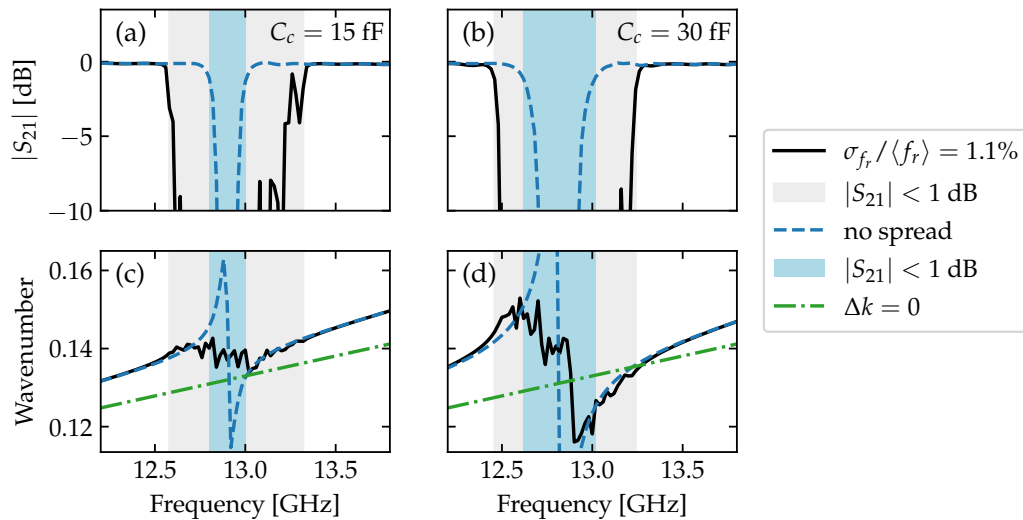


Fig. 3.25.: Effect of resonance-frequency spread on resonant phase matching in an RPM-TWPA with 250 resonators. Shown is the small-signal transmission coefficient and wavenumber in the vicinity of the resonant dispersive feature, both without spread (dashed blue lines) and with a representative example with a spread of 1.1% (solid black lines). Phase matching is achieved where the dashed-blue or solid-black lines intersect with the dash-dotted green lines. Without spread, this condition is met outside the resonance dip (blue-shaded areas) for both coupling strengths set by C_c . Panels (c) and (d), implying phase matching without severe pump depletion. With resonance-frequency spread, resonance dips are widened (gray-shaded areas). **(c)** Lower coupling strength: Phase matching is not possible. **(d)** Stronger coupling: Phase matching is possible without severe pump depletion even in the presence of spread. Small-signal single-tone simulations were performed with a current of 0.1 μA and otherwise as stated in Fig. 3.7.

resonators. For example, using $\lambda/4$ transmission-line resonators a resonance-frequency spread $< 0.1\%$ has been achieved [146]. This approach also comes at the cost of larger footprint, c.f. [111]. Nevertheless, it is worth mentioning that 3WM, unlike 4WM, requires only half the length of RPM transmission-line resonators due to the pump frequency being twice as high.

3.5.2 Comparison of PCV and RPM regarding parameter spread

Figure 3.26 allows a comparison of the RPM-TWPA and the PCV-TWPA concerning their sensitivity to the spread of inductances, critical currents of Josephson junctions, and capacitances. While the RPM-TWPA is found to be vulnerable to resonance-frequency spread, it seems substantially more tolerant to the spreads of the other circuit parameters, Fig. 3.26a–c, because they do not directly affect phase matching.

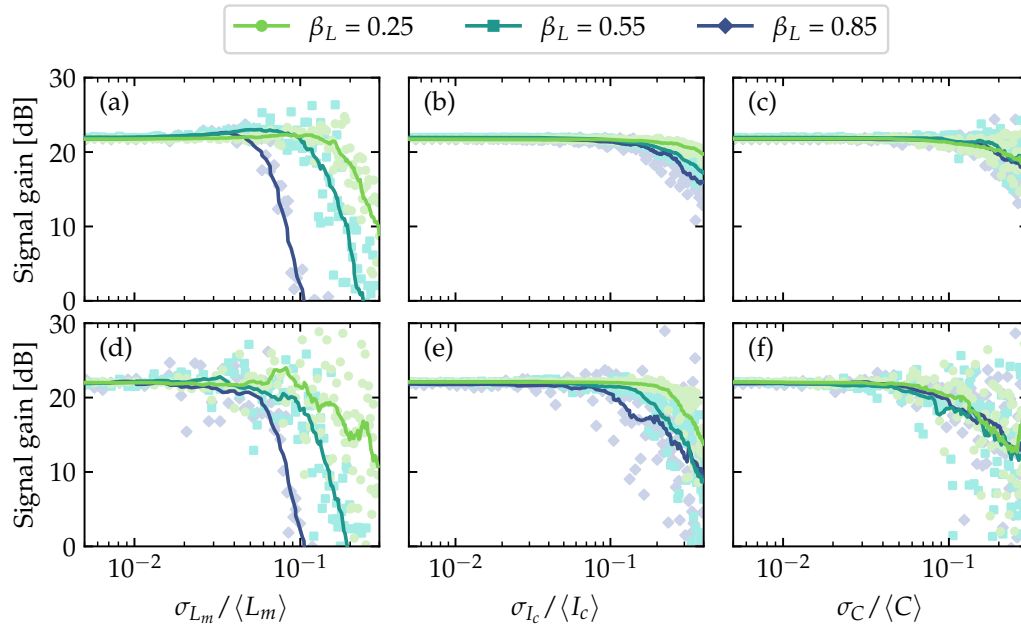


Fig. 3.26.: Sensitivity analysis of parameter spread of inductances, critical currents, and capacitances on the performance of the RPM-TWPA, (a)–(c), and the PCV-TWPA, (d)–(f). The parameter in each panel is altered following a normal distribution, while the rest of the parameters remain unchanged. Capacitance spread refers to the variation of C_n , C_J , and C_c , while C_r is kept constant in the case of RPM, analyzed separately in Fig. 3.24. The maximum gain obtained within a pump frequency range of 12.98–13.1 GHz is plotted. Solid lines depict moving averages computed from 20 consecutive samples, highlighting trends. The gain decreases with increasing parameter spread, accompanied by microwave reflections that manifest as divergent gain values exceeding the zero-spread gain. Simulation parameters in (a)–(c): Signal frequency of 7 GHz, and pump currents of 6.59, 2.8, 1.8 μA for $\beta_L = 0.25, 0.55, 0.85$, respectively. (d)–(e): Signal frequency of 7.7 GHz, and pump currents of 3.84, 1.63, 1.07 μA for $\beta_L = 0.25, 0.55, 0.85$, respectively.

Despite the fact that phase matching in the PCV-TWPA is affected by these parameters, it is only slightly more sensitive to their spread than the RPM-TWPA, compare Fig. 3.26d–f with a–c. In PCV, phase matching is provided by the stopband created due to the spatially-periodic shape of the local values of the phase velocity and the characteristic impedance, see Sec. 3.2.3. Stochastic variations of circuit parameters alter these periodic shapes and, consequently, affect phase matching. On the other hand, one period comprises many unit cells, such that small stochastic variations are averaged out on the scale of one or more periods. This is in stark contrast to RPM, where each individual resonator has a direct effect on phase matching. Even a single outlier is sufficient to cause severe pump depletion because of its resonance dip.

Typical parameter spread values reported in the literature are $< 2\%$ for capacitances and inductances [143, 144, 145], and 1–8% for the critical currents of both Nb-trilayer and Al shadow-evaporated Josephson junctions [147, 148, 149, 150, 151]. Our sensitivity analysis shows that these values are rather uncritical for both TWPA concepts, leaving the resonance-frequency spread aside. Although the PCV-TWPA is roughly 2–4 times more sensitive to parameter spread than the RPM-TWPA, we may conclude that PCV does not possess a critical circuit element such as the resonators in RPM.

3.5.3 Spread of critical currents, capacitances, and inductances

Enhanced microwave reflections due to parameter spread

The most substantial effect among the parameters $L_m, I_c, C_{0,c,J}$ stems from inductance spread, because L_m directly influences flux biasing, inductance, and rf-SQUID nonlinearity, thereby affecting local impedance and wavenumber. Apart from reducing the signal gain by affected phase matching, these inhomogeneities in the circuit cause increased microwave reflections [141] and gain ripple, c.f. Ref. [98] for a Josephson-junction-based TWPA. Constructively or destructively interfering microwave reflections correspond to peaks above or dips below the zero-spread gain value. This diverging behaviour of the gain values at increasing spread is most pronounced in Fig. 3.26a,d, but they are also observable in the other panels for greater spread values.

Additionally, it can be observed from Fig. 3.26a,d that the tolerance to inductance spread can be greatly increased by decreasing parameter $\beta_L \propto L_m I_c$, which represents a degree of freedom in rf-SQUID-based TWPAs. This parameter determines the level of nonlinearity and flux-tunability of the SQUID (c.f. Fig 2.4), explaining the different sensitivities to inductance spread. For the simulations, we change β_L via I_c , keeping L_m constant. Because the exponential gain coefficient of the three-mode model (Eq. 2.96) is proportional to β_L , the pump current is adjusted accordingly to achieve similar zero-spread gain for all β_L values (see caption of Fig. 3.26).

Hysteresis in rf-SQUIDs due to parameter spread

The relatively sharp drop of the signal gain with inductance spread, starting at a spread of 5% for $\beta_L = 0.85$, may have a different explanation than its influences on gain coefficient or phase matching. Instead, the sharp drop may be attributed to the presence of a single or few hysteretical rf-SQUIDs, i.e., having $\beta_L > 1$. Operating a hysteretical rf-SQUID near its hysteresis, each period ω^{-1} of an alternating current

(e.g. the pump) injected into the SQUID causes one circulation around the hysteresis. This circulation is equivalent to inserting and expelling one flux quantum Φ_0 per cycle, and the corresponding switching energy is dissipated on the rate of the driving AC current. Therefore, the presence of even a single hysteretic rf-SQUID may cause severe dissipation in the TWPA circuit, which explains the sharp drop of gain with increased inductance spread.

Note that the same sharp drop is not observed for critical-current spread, although $\beta_L \propto L_m I_c$. However, while I_c primarily influences β_L (potentially increasing it above unity), L_m additionally alters the flux bias of a current-biased rf-SQUID, $\Phi_{\text{ext,dc}} = L_m I_{\text{dc}}$. Having an inductance L_m sufficiently larger than its target value thus has two effects: First, it causes hysteresis, and second, it shifts the bias point closer to the hysteresis, potentially causing dissipation.

Robustness to critical-current spread

It is intriguing that the rf-SQUID-based TWPA is quite insensitive to critical-current spread. Particularly for low β_L , the curves in Fig. 3.26b,e are rather flat up to spreads of 10–20%. This is in stark contrast to the tolerable spread value of 3% for the SNAIL-based reversed-Kerr TWPA analyzed theoretically in Ref. [139], or to a spread of 5% causing drastically increased reflections and ± 5 dB gain ripple in the gain profile of the Floquet-TWPA based on Josephson junctions [98]. Moreover, in both SNAIL-based or Josephson-junction-based TWPAs, the element with the smallest critical current places an absolute limit to the range of usable pump currents, above which the device switches to the normal-conducting state.

To show why the rf-SQUID-based TWPA is relatively insensitive to critical-current spread, we propagate the variances $\sigma_{L_m}^2$ and $\sigma_{I_c}^2$ to the variance of the rf-SQUID inductance,

$$\sigma_{L_{\text{SQ}}}^2 \approx \left| \frac{\partial L_{\text{SQ}}}{\partial L_m} \right|^2 \sigma_{L_m}^2 + \left| \frac{\partial L_{\text{SQ}}}{\partial I_c} \right|^2 \sigma_{I_c}^2. \quad (3.64)$$

The derivatives are calculated from $L_{\text{SQ}} = L_m / (1 + \beta_L \cos \phi_{\text{dc}})$ (Eq. 2.24), where both ϕ_{dc} and $\beta_L = L_m I_c / \varphi_0$ depend on I_c and L_m . The dependence of ϕ_{dc} is given by the transcendental equation (Eq. 2.23),

$$\phi_{\text{dc}} + \frac{L_m I_c}{\varphi_0} \sin \phi_{\text{dc}} = \frac{L_m I_{\text{dc}}}{\varphi_0}, \quad (3.65)$$

assuming that the rf-SQUID is biased with current I_{dc} , such that $\phi_{ext,dc} = L_m I_{dc} / \varphi_0$. By taking into account the implicit derivatives of ϕ_{dc} ,

$$\frac{\partial \phi_{dc}}{\partial L_m} = \frac{I_{dc} - I_c \sin \phi_{dc}}{\varphi_0 (1 + \beta_L \cos \phi_{dc})} = \frac{1}{L_m} \frac{\phi_{ext} - \beta_L \sin \phi_{dc}}{1 + \beta_L \cos \phi_{dc}}, \quad (3.66)$$

$$\frac{\partial \phi_{dc}}{\partial I_c} = -\frac{1}{I_c} \frac{\beta_L \sin \phi_{dc}}{1 + \beta_L \cos \phi_{dc}}, \quad (3.67)$$

we get the derivatives of the SQUID inductance,

$$\frac{\partial L_{SQ}}{\partial L_m} = \frac{1 + \beta_L \frac{\phi_{ext} - \beta_L \sin \phi_{dc}}{1 + \beta_L \cos \phi_{dc}} \sin \phi_{dc}}{(1 + \beta_L \cos \phi_{dc})^2} \quad (3.68)$$

and

$$\frac{\partial L_{SQ}}{\partial I_c} = -\frac{L_m \beta_L}{I_c} \frac{\cos \phi_{dc} + \frac{\beta_L \sin^2 \phi_{dc}}{1 + \beta_L \cos \phi_{dc}}}{(1 + \beta_L \cos \phi_{dc})^2}. \quad (3.69)$$

Equation 3.69 is zero at $\phi_{dc} \approx \beta_L + \pi/2$, approximately at the same point where the quadratic-nonlinearity coefficient β is maximal. At this point, the rf-SQUID is first-order insensitive to critical-current spread. For a flux bias at the Kerr-free point, $\phi_{dc} = \pi/2$, the normalized standard deviation of the rf-SQUID inductance simplifies to

$$\frac{\sigma_{L_{SQ}}}{\langle L_{SQ} \rangle} = \sqrt{\left(1 + \frac{\pi}{2} \beta_L\right)^2 \frac{\sigma_{L_m}^2}{\langle L_m \rangle^2} + \beta_L^4 \frac{\sigma_{I_c}^2}{\langle I_c \rangle^2}}. \quad (3.70)$$

Evaluating this equation for $\beta_L = 0.25$ and assuming an inductance spread of 1% and a critical-current spread of 5%, the resulting spread of the rf-SQUID inductance is 1.4%, and is determined almost entirely (98%) by inductance spread. At this flux bias, the inductance of the Josephson inductance goes to infinity, and the SQUID inductance is $L_{SQ} = L_m$. Hence, the Josephson junction contributes only to nonlinearity, but not to the linear inductance. This illustratively explains, in a simplified picture, the low sensitivity to critical-current spread. Therefore, the low sensitivity of critical-current spread in rf-SQUID-based TWPAs originates predominantly in the choice of flux bias for zero Kerr or maximized quadratic-nonlinearity.

3.5.4 Practical implications and experimental evidence

In summary, the sensitivity of two dispersion-engineered rf-SQUID-based TWPA concepts to the spread of critical currents of Josephson junctions, inductances, and capacitances has been investigated with the help of circuit simulations. The performance of a TWPA utilizing RPM is highly sensitive to resonance-frequency

spread. Its sensitivity can be decreased by increasing their coupling strength. A TWPA using PCV does not have a critical circuit element such as the resonators, even though PCV is slightly more sensitive to the spread of the other circuit parameters. Among these parameters, the inductance spread plays the largest role in affecting amplification performance of both TWPA concepts. However, the literature indicates that typical fabrication processes yield inductance spreads that are safely tolerable, particularly for low β_L values. Remarkably, rf-SQUID-based 3WM TWPAs are rather insensitive to Josephson-junction critical-current spread, in great contrast to TWPAs based on other nonlinear inductive elements. By using rf-SQUIDs biased close to their Kerr-free point, the robustness of the circuit benefits from the precision at which inductors can be fabricated [143, 144], especially when compared to Josephson junctions.

To optimize and practically realize a TWPA concept, it is crucial to understand the sensitivity of the circuit to expectable parameter spread. At PTB, the present work has paved the way towards the experimental implementation of PCV-TWPAs rather than RPM-TWPAs, as detailed in Chapter 4. In this implementation, a critical-current spread of $> 20\%$ was found (refer to Appendix A.2.1, Fig. A.4d), the origin of which is not yet understood. Thus, the convincing amplifier performance, despite this enormous spread, gives evidence to the high robustness of rf-SQUID-based 3WM TWPAs against critical-current spread as predicted in this simulation study.

Experimental realization and characterization of a TWPA with multi-periodic capacitance variation

This chapter is dedicated to the implementation and characterization of an rf-SQUID based three-wave-mixing TWPA with multi-periodic capacitance variation (mPCV) as described in Chapter 3. The development of TWPAs at PTB has gone through multiple design iterations since the seminal paper of Zorin [57]. Early TWPA generations suffered from multiple issues which had to be tackled during the course of my work. Major improvements include more accurate monitoring and control of circuit parameters, proper microwave design to prevent resonant modes within the signal band, calibration techniques for scattering-parameter measurements in cryogenic environments, and better filtering in the setup avoiding frequency-conversion loss induced by low-frequency noise. However, the main improvement resulting in a well-performing TWPA was proper dispersion engineering with circuit parameters fine-tuned in circuit simulations. The results presented in this chapter correspond to the third mPCV-TWPA design iteration, which is the first TWPA design at PTB exhibiting convincing performance¹.

This chapter is structured in four parts. First, I present the sample design and circuit model, and discuss precharacterization and circuit parameter extraction. The second part is dedicated to the issue of frequency-conversion loss, followed by the third part, the detailed characterization of parametric amplification. Finally, the noise characterization is discussed. The chapter concludes with a comparison to other state-of-the-art TWPAs.

¹Early results, such as the signal gain of ca. 11 dB in an rf-SQUID-based TWPA without any dispersion engineering, reported in Ref. [75], could not be reproduced in later samples. At that time, the gain was measured as a pump-on-pump-off ratio, which does not take into account any insertion loss, such as frequency-conversion loss (Sec. 4.4). This loss mechanism exhibits a pump-power dependence, which can misinterpreted as signal gain, as explained in Sec. 4.4, when an absolute transmission calibration is not applied.

4.1 Sample design and fabrication

Table 4.1 gives an overview of the targeted circuit parameters for the sample design of the TWPA with multi-periodic capacitance variation. In comparison to the circuit parameters studied in chapter 3, the screening parameter β_L , the inductance L_m , and consequently the mean capacitance $\bar{C}$ were reduced to achieve larger saturation powers while keeping the impedance at $\bar{Z} = \sqrt{(L_m + L_f)/\bar{C}} = 50 \Omega$. We targeted a signal band centered around 6 GHz, having a bandwidth of ca. 4 GHz (one octave), which sets the targeted pump frequency to ca. 12 GHz. To achieve phase-matching, the pump frequency should be placed slightly above the upper edge of the first stopband. Therefore, a period of $m = 28 = 4 \times 7$ unit cells is chosen, resulting in a position of the first stopband at $1/2m\sqrt{L_m\bar{C}} = 11.2$ GHz (see Sec. 3.2.4). Since the preceding simulations did not account for dielectric loss, different lengths of the circuit were tried, including 1904 unit cells (sample A) and 2352 unit cells (samples B and C), to compensate for loss and other effects reducing the gain. Apart from the circuit length, samples A, B, and C differ only in the value of the capacitance variation depth ζ_1 . Sample B showed the best performance and is addressed in the remainder of the text if not stated otherwise.

An overview of the sample design of the TWPA is given in Fig. 4.1. The basic building block of the circuit is a pi-type unit cell of length $a = 52 \mu\text{m}$ formed by an rf-SQUID and two plate capacitors connected to the ground planes of the embedding coplanar waveguide (CPW). Multiperiodic capacitance variation according to the scheme discussed in Sec. 3.2.4 is realized by varying the ground capacitances of each period m between three steps C_{01}, C_{02}, C_{03} , yielding capacitance variation depths ζ_1, ζ_2 according to Eqs. 3.31 and 3.32. Figure 4.1d shows the unit cells with varied

Tab. 4.1.: Targeted circuit parameters

	Sample A	Sample B	Sample C
Meander inductance		$L_m = 60 \text{ pH}$	
Passive inductance		$L_f = 20 \text{ pH}$	
Critical current		$I_c = 1.3 \mu\text{A}$	
Screening parameter		$\beta_L = 0.24$	
Ground capacitance		$\bar{C} = 32 \text{ fF}$	
Variation depth 1	$\zeta_1 = 0.14$	$\zeta_1 = 0.17$	$\zeta_1 = 0.20$
Variation depth 2		$\zeta_2 = 0.85$	
Unit cells per period		$m = 28$	
Number of unit cells	$N = 1904$	$N = 2352$	$N = 2352$

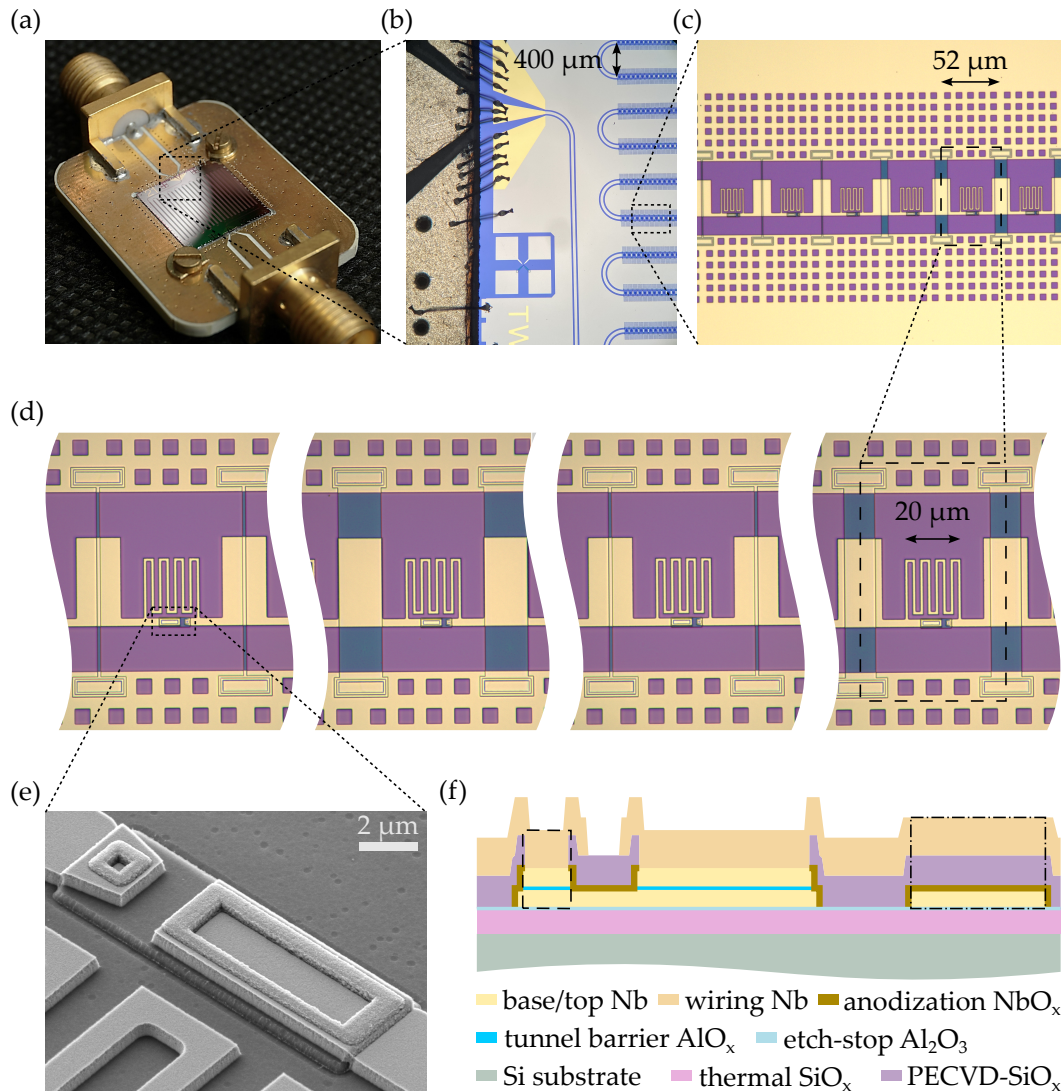


Fig. 4.1.: Implementation of the TWPA circuit with multiperiodic capacitance variation. (a) TWPA chip wirebonded to a printed circuit board (PCB). The size of the chip is 9×9 mm. (b)-(d) Optical micrographs of the TWPA sample. The TWPA circuit is an array of 2352 unit cells embedded in a meandering coplanar waveguide (CPW), arranged in 21 meanders of 4 sections each. One section consists of $m = 4 \times 7 = 28$ cells in four groups of seven nominally identical cells. The dashed rectangle in (c) and (d) marks one cell. Each cell consists of an rf-SQUID and two plate capacitors between center conductor and ground plane (plate capacitors are shared among adjacent cells). The width of the capacitor plates is varied between three values in the sequence depicted in (d). Rectangular holes in the ground plane are flux traps to avoid moving vortices. The rf-SQUID is realized as a quasi-one-junction SQUID, possessing a small and a large Josephson junction in parallel with a meander inductor. Panel (e) shows a scanning electron micrograph of the two Josephson junctions. The small Josephson junction with size $1 \times 1 \mu\text{m}$ is the active junction dominating the quantum interferences of the SQUID, while the large Josephson junction is used as a via connecting the small junction back to the wiring metal layer. Panel (f) schematizes the layer stack (not drawn to scale). The dashed rectangle represents a Josephson junction, formed by a Nb/Al-AlO_x/Nb trilayer. The dashdotted rectangle depicts a plate capacitor with PECVD-SiO_x as dielectric, and base-Nb and wiring-Nb as electrodes. Scanning electron micrograph by Jan Blohm and Victor Gaydamachenko.

ground capacitor geometries, which are arranged in four groups of seven nominally identical cells in the sequence $7 \times C_{01}, 7 \times C_{02}, 7 \times C_{01}, 7 \times C_{03}$.

Capacitances are realized as plate capacitors, formed by a PECVD-SiO_x dielectric layer sandwiched between electrodes of the base and wiring Nb layers (Fig. 4.1f). In the design, stray capacitances and fringing effects are neglected, assuming that the capacitance of one unit cell is proportional to the overlap area of the capacitor plates. The width of the bottom plate (base Nb layer, dark green areas in Fig. 4.1d) is varied, while the geometry of the top plate (wiring Nb layer, ocher-colored areas) remains identical to avoid altering the geometrical inductance of the cell. It turned out later that the stray capacitance of the structures in the center conductor to the ground planes is on the order of 7 fF per unit cell, which is on the same order as the smallest unit-cell capacitance value, C_{01} . Consequently, neglecting the stray capacitance underestimates the mean capacitance $\bar{C}$ by ca. 30%, which shifts the position of the stopbands to lower frequencies than planned. Future design iterations will contain plate capacitor geometries where stray capacitance and fringing effects are accounted for by the use of 3D electromagnetic simulations.

All cells have nominally identical rf-SQUIDs, consisting of a meander inductor with inductance L_m in parallel to two Josephson junctions. Because the performance of the TWPA depends strongly on the value of L_m , we optimized the geometry of the meander inductor using the quasistatic 3D electromagnetic simulation software Ansys Q3D Extractor® to be close to its target inductance. A 3 μm wide strip connects rf-SQUID and plate capacitor and increases the cell inductance by L_f . The rf-SQUID is designed as a quasi-one-junction SQUID (Sec. 2.1.3), where the small Josephson junction with critical current I_c is the active junction, dominating the quantum interferences in the SQUID and causing nonlinearity, and the large Josephson junction with critical current $I_{c,L} = 14I_c$ serves the purpose of connecting the base Nb electrode of the small junction to the wiring Nb layer. This large junction is required because the fabrication process does not contain a lithography step for vias between the base and wiring Nb layers. Figure 4.1e,f shows a scanning electron micrograph of the two Josephson junctions and the corresponding layer scheme.

The whole circuit consists of 21 arrays with $4m$ unit cells each. These arrays are connected in a meandering manner by CPW segments, which are designed to have a characteristic impedance of 50 Ω. Both ends of the CPW are wirebonded to a grounded coplanar waveguide (GCPW) on a printed circuit board (PCB), see Fig. 4.1a,b, connecting the sample to the measurement setup. The ground planes of the sample and the PCB are connected by bondwires along the circumference of the sample to prevent resonant slotline modes by unequal ground potentials. Likewise, on the sample the same purpose is fulfilled by the lower electrodes of

the plate capacitors, which connect the two CPW ground planes and equalize their potentials. Rectangular holes in the ground planes serve as vortex pinning centers, often called flux traps. Furthermore, we designed the rf-SQUIDs and plate capacitors as narrow superconducting strips with widths $\leq 16 \mu\text{m}$ to increase the probability that vortices are fully expelled [152]. These measures prevent vortices from locally altering the flux-sensitive operation points of the SQUIDs.

The samples were fabricated in the PTB cleanroom facilities on $380 \mu\text{m}$ thick 3 inch Silicon wafers with 200 nm thermal SiO_x . Josephson junctions were made in Nb/Al- AlO_x /Nb trilayer technology with a targeted critical-current density of $J_c = 1.3 \mu\text{A}/\mu\text{m}^2$. One additional Nb layer is used for wiring with a PECVD- SiO_x layer as dielectric. All lithography steps were done using electron beam lithography. The technology is described in Ref. [153]. Victor Gaydamachenko created all layouts and fabricated all samples presented in this chapter.

4.2 Circuit model and linear characterization

This section describes a series of precharacterization microwave measurements. The goal of this procedure is two-fold: First, it allows to monitor the fabrication process and to adjust future designs or fabrication recipes accordingly. Second, extracting a multitude of lumped circuit parameters allows to rigorously model and further analyze the devices.

All precharacterization microwave measurements are carried out in liquid helium (4.2 K) and require special calibration techniques suitable for cryogenic environments. Measurements are performed on passive monitoring samples (Sec. 4.2.3) and on TWPA samples (Sec. 4.2.4). The former yield multiple linear circuit parameters such as capacitances and inductances, and allow estimating dielectric loss and quasiparticle loss at 4.2 K.

Some further complementary measurements are described in Appendix A.2. These include DC measurements, carried out both at room temperature and in liquid helium, to characterize long and narrow Nb film strips as well as test structures of single Josephson junctions. The former give insight on the expected kinetic inductance and surface resistance of the Nb film. The latter yield characteristics such as the critical-current density and the subgap resistance of the Josephson junctions, having influence on the nonlinearity and the microwave loss of the TWPA circuit, respectively (Sec. 4.2.1). The critical-current spread across the whole wafer is also obtained.

4.2.1 Detailed circuit model

The wavelength of the second harmonic of the pump is much larger than the length of one unit cell, $\lambda \approx ma \gg a$. Hence, we can model the circuit elements of a unit cell as lumped elements, adequate for all frequencies involved in parametric processes up to at least second order (including frequencies such as f_{p+s} , f_{p+i} , f_{2p} , etc., see Sec. 2.4.1). Figure 4.2 illustrates the circuit model of the TWPA and its relation to the sample design. The circuit model used for further investigations is simplified and makes use of several approximations, which are explained in the following.

The model consists of cascaded two-ports, each modeling either a TWPA unit cell, or transmission-line segments, or bondwires connecting the transmission line on the chip with that on the PCB. The resulting circuit model is one-dimensional, and neglects higher-dimensional effects, such as crosstalk between adjacent arrays of unit cells, or coupling to external fields like cavity modes in the sample package. The major advantage of this approach is that the circuit can be represented by cascaded transfer matrices, which we will use extensively for linear analysis.

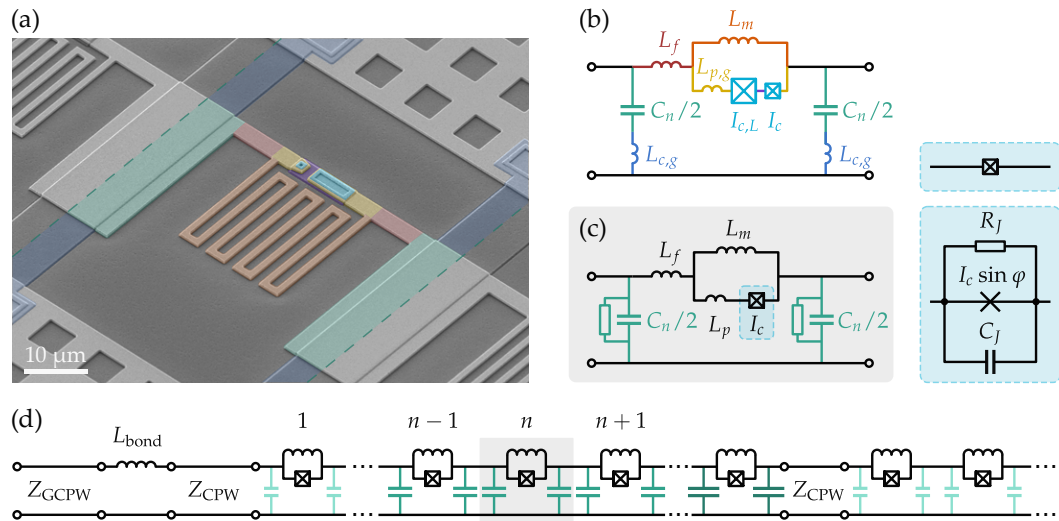


Fig. 4.2.: TWPA circuit model. **(a)** False-color scanning-electron micrograph of one unit cell, and **(b)**, corresponding circuit representation. **(c)** Circuit model of one unit cell with index $n = 1 \dots N$, used for further analysis, where $L_{c,g}$ is neglected, $L_{p,g}$ and the large junction are combined to L_p , and parallel resistors account for the dielectric loss of the plate capacitors. The small Josephson junction is represented by an RCSJ model. **(d)** The circuit model of the whole circuit also accounts for bondwire inductances, CPW segments on the chip, and GCPW segments on the PCB. All unit cells have identical circuit parameters, except the capacitors C_n , which are varied periodically. Scanning electron micrograph by Jan Blohm and Victor Gaydamachenko.

Josephson junction

The Josephson junction with critical current I_c is represented by a resistively and capacitively shunted junction (RCSJ) model, having junction capacitance C_J and resistance R_J in parallel to an ideal Josephson junction with $I = I_c \sin(\varphi)$, where φ is the phase difference across the junction. The Josephson junction in an rf-SQUID operates always in the zero-voltage state, since it is shunted by a superconducting inductor. Therefore, R_J is equal to the subgap resistance R_{sg} , see Sec. A.2.1.

rf-SQUID

The model of the rf-SQUID takes into account nonidealities such as the large Josephson junction used as a via, having critical current $I_{c,L}$, and the parasitic inductance $L_{p,g}$ in the Josephson junction branch of the SQUID. Such a SQUID is described in Sec. 2.1.3. Although the large Josephson junction does not introduce noticeable nonlinearity, it adds considerable linear inductance $L_{J0,L} = \varphi_0 / I_{c,L}$ to the SQUID loop. These nonidealities are accounted for by the parasitic inductance $L_p = L_{p,g} + L_{J0,L}$, which combines the geometric inductance of the branch with the linear inductance of the large via junction. The total inductance of the SQUID loop

is given by $L_{\text{loop}} = L_m + L_p$, with the meander inductance L_m . Adapting the model of the quasi-one-junction SQUID with parasitic inductances, Eqs. 2.36–2.41, the system of equations describing the inductance of an rf-SQUID is

$$L_{\text{SQ}}(\phi_{\text{dc}}) = L_m \frac{(1 + \alpha \beta_L \cos \phi_{\text{dc}})}{\beta_L \cos \phi_{\text{dc}}}, \quad (4.1)$$

with the phase ϕ_{dc} given by the transcendental equation

$$\phi_{\text{dc}} + \beta_L \sin \phi_{\text{dc}} = \phi_{\text{ext,dc}}, \quad (4.2)$$

and the definitions for the nonideality coefficient

$$\alpha = \frac{L_p}{L_m + L_p} \quad (4.3)$$

and the rf-SQUID screening parameter

$$\beta_L = \frac{2\pi(L_m + L_p)I_c}{\Phi_0}. \quad (4.4)$$

Considering the linear inductance L_f of the connecting segment between two adjacent SQUIDs, the inductance of one unit cell is

$$L_{\text{cell}}(\phi_{\text{dc}}) = L_f + L_{\text{SQ}}(\phi_{\text{dc}}). \quad (4.5)$$

The corresponding nonlinearity coefficients of second and third order are given by Eqs. 2.40 and 2.41, and read

$$\beta = \frac{p\beta_L(1 - \alpha_f)}{2} \frac{\sin \phi_{\text{dc}}}{(1 + \beta_L \cos \phi_{\text{dc}})(1 + \alpha_f \beta_L \cos \phi_{\text{dc}})^2}, \quad (4.6)$$

$$\gamma = \frac{p^2\beta_L(1 - \alpha_f)}{6} \frac{(1 + \alpha_f \beta_L \cos \phi_{\text{dc}}) \cos \phi_{\text{dc}} + 3\alpha_f \beta_L \sin^2 \phi_{\text{dc}}}{(1 + \beta_L \cos \phi_{\text{dc}})(1 + \alpha_f \beta_L \cos \phi_{\text{dc}})^4}, \quad (4.7)$$

with $\alpha_f = (\alpha L_m + L_f)/(L_m + L_f)$, and $p = L_m/(L_m + L_f)$. The linear inductance L_f reduces the participation ratio of the nonlinear inductance relative to the total unit-cell inductance L_{cell} , effectively diluting the nonlinearity of the cell inductance compared to that of an rf-SQUID alone.

Meander inductor

The meander inductor is a superconducting strip, which is much longer than it is wide. Therefore, the surface impedance $Z_s = R_s + j\omega L_s$, stemming from the

penetration of fields into the surface of the strip, contributes substantially to the impedance of the strip, see Appendix A.2.2. The inductance has three components: the inductances due to the magnetic field both outside and inside the conductor (referred to as external and internal geometric inductance), and the kinetic inductance arising from the inertia of the Cooper pairs. The latter two form the surface inductance L_s . In the model, the meander inductance contains all three contributions, $L_m = L_{m,\text{ext}} + L_s$.

Quasiparticle loss

At 4.2 K, quasiparticle loss of the meander, represented as the surface resistance R_s , is one of the major sources of loss in the TWPA. For convenience, we combine all three manifestations of quasiparticle loss, which are R_s and the subgap resistances of the two Josephson junctions. The subgap resistances of the two Josephson junctions can be combined to an effective subgap resistance $R_{\text{sg,eff}} = R_{\text{sg}} / (1 + I_c / I_{c,L})$ in the RCSJ model of the small junction. Under the condition $R_{\text{sg}} \ll \omega \varphi_0 / I_c$ this leads to an effective subgap resistance

$$R_{\text{sg,eff}}^{-1} = R_{\text{sg}}^{-1} \left(1 + \frac{I_c}{I_{c,L}} \right) + \frac{R_s}{\omega^2 L_{\text{SQ}}^2} \quad (4.8)$$

in the RCSJ model of the small junction. Roughly, $R_s \propto \omega^2$, see Appendix A.2.2, so $R_{\text{sg,eff}}$ is frequency-independent in a first approximation and can be modeled as a constant resistor. As quasiparticles freeze out at low temperatures, their attributed losses are negligible compared to dielectric loss for temperatures lower than ~ 2 K, see temperature dependences in Appendix A.2.

Plate capacitors

The dielectric loss of the plate capacitors is accounted for via the loss tangent $\tan \delta$ and is the dominant source of loss in the TWPA at low temperatures < 2 K (see Sec. 2.2.3). To account for loss, the capacitance C_n can be modeled as a complex capacitance $C_n(1 - j \tan \delta)$, equivalent to adding a resistor

$$R_{C,n} = 1 / (\omega C_n \tan \delta) \quad (4.9)$$

in parallel to C_n , which indicates a frequency dependence $\propto \omega$. The lower plate of the plate capacitors is a strip with a considerable parasitic inductance $\tilde{L}_{c,g} = 16\text{--}38$ pH between the two ground planes, determined by electromagnetic simulations and depending on the width of the lower plate. Large Josephson junctions with a critical current of ca. 85 μA serve as vias and add inductance $L_{J,g} =$

$\varphi_0/85 \mu\text{A} \approx 4 \text{ pH}$. Due to the parallel arrangement of the ground planes, the effective series inductance, as sketched in Fig. 4.2b, is $L_{c,g} = (\tilde{L}_{c,g} + L_{J,g})/4$. Despite the large inductance values, $\omega L_{c,g} \ll 1/(\omega C_n)$ for relevant frequencies, so we neglect $L_{c,g}$ in the further analysis. The error in impedance at 20 GHz is smaller than 1%.

4.2.2 Cascaded-transfer-matrices model

A convenient way to model the passive monitoring samples is the cascaded-transfer-matrices (CTM) method, a standard approach in microwave engineering [85]. To model the linear behavior of the TWPA, the same model is used. The circuit is a one-dimensional cascade of $\tilde{N}$ two-ports, each represented by a transfer matrix T_ν with the index $\nu = 1 \dots \tilde{N}$, and $\tilde{N} > N$, also taking into account segments of transmission lines, and bond wires. A unit cell with index $n = 1 \dots N$ is described by

$$T_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} = \begin{pmatrix} 1 + Y_{2,n}/Y_3 & 1/Y_3 \\ Y_{1,n} + Y_{2,n} + Y_{1,n}Y_3/Y_{2,n} & 1 + Y_{1,n}/Y_3 \end{pmatrix}, \quad (4.10)$$

with the shunt admittances $Y_{1,n}, Y_{2,n}$ and the series admittance Y_3 of a pi-type two-port [85], yielding

$$a_n = 1 - \frac{1}{2} \frac{\omega^2 L_{\text{cell}} C_n (1 - j \tan \delta)}{1 + j\omega L_{\text{cell}}/R_J - \omega^2 L_{\text{cell}} C_J}, \quad (4.11)$$

$$b_n = \frac{j\omega L_{\text{cell}}}{1 + j\omega L_{\text{cell}}/R_J - \omega^2 L_{\text{cell}} C_J}, \quad (4.12)$$

$$c_n = j\omega C_n (1 - j \tan \delta) - \frac{1}{4} \frac{j\omega^3 L_{\text{cell}} (C_n (1 - j \tan \delta))^2}{1 + j\omega L_{\text{cell}}/R_J - \omega^2 L_{\text{cell}} C_J}, \quad (4.13)$$

$$d_n = a_n. \quad (4.14)$$

Here, $L_{\text{cell}} = L_{\text{cell}}(\phi_{\text{dc}}) = L_f + L_{\text{SQ}}(\phi_{\text{dc}})$ (Eq. 4.5) for TWPAs, or $L_{\text{cell}} = L_f + L_m$ in the case of passive monitoring samples. The transfer matrix of a lossless transmission line, e.g. a CPW having characteristic impedance $Z_{\text{CPW}} \equiv 1/Y_{\text{CPW}}$, length ℓ , and effective permittivity ϵ_{eff} yielding propagation constant $\beta_{\text{CPW}} = \sqrt{\epsilon_{\text{CPW}}} \omega/c$, where c is the speed of light, is [85]

$$T_{\text{CPW}} = \begin{pmatrix} \cos \beta_{\text{CPW}} \ell & j Z_{\text{CPW}} \sin \beta_{\text{CPW}} \ell \\ j Y_{\text{CPW}} \sin \beta_{\text{CPW}} \ell & \cos \beta_{\text{CPW}} \ell \end{pmatrix}, \quad (4.15)$$

and analogous for a GCPW with corresponding indices. The transfer matrix

$$T_{\text{bond}} = \begin{pmatrix} 1 & j\omega L_{\text{bond}} \\ 0 & 1 \end{pmatrix} \quad (4.16)$$

describes bond wires. The circuit includes $N/4m$ arrays of $4m$ unit cells each, plus a CPW 180° turn connecting adjacent arrays, and CPW and GCPW segments at the input and output of the device with bond wires between. The corresponding $\tilde{N}$ transfer matrices are then cascaded to a system transfer matrix

$$T_{\tilde{N}} = \begin{pmatrix} a_{\tilde{N}} & b_{\tilde{N}} \\ c_{\tilde{N}} & d_{\tilde{N}} \end{pmatrix} = T_{\text{GCPW}} T_{\text{bond}} T_{\text{CPW}}^{\text{in}} (T_{\text{array}} T_{\text{CPW}}^{\text{turn}})^{\frac{N}{4m}-1} T_{\text{array}} T_{\text{CPW}}^{\text{out}} T_{\text{bond}} T_{\text{GCPW}}, \quad (4.17)$$

with the transfer matrix of one array,

$$T_{\text{array}} = \left(T_{01}^{\frac{m}{4}} T_{02}^{\frac{m}{4}} T_{01}^{\frac{m}{4}} T_{03}^{\frac{m}{4}} \right)^4, \quad (4.18)$$

where the indices 01, 02, and 03 refer to the respective ground capacitor variant C_n (see Fig. 3.10). The superscripts 'in', 'out', and 'turn' denote CPW segments of corresponding lengths, and all matrix exponents are integer numbers. The system transfer matrix can then be transformed into a scattering matrix for a given reference impedance Z_0 [85]. The input reflection coefficient S_{11} and forward transmission coefficient S_{21} are given by the following expressions:

$$S_{11} = \frac{a_{\tilde{N}} + b_{\tilde{N}}/Z_0 - c_{\tilde{N}}Z_0 - d_{\tilde{N}}}{a_{\tilde{N}} + b_{\tilde{N}}/Z_0 + c_{\tilde{N}}Z_0 + d_{\tilde{N}}}, \quad (4.19)$$

$$S_{21} = \frac{2}{a_{\tilde{N}} + b_{\tilde{N}}/Z_0 + c_{\tilde{N}}Z_0 + d_{\tilde{N}}}. \quad (4.20)$$

Since the CTM model describes the linearized circuit in the small-signal case (pump off), the circuit is reciprocal, $S_{12} = S_{21}$. Moreover, $S_{22} \approx S_{11}$.

4.2.3 Passive mPCV-TWPA monitoring samples

Figure 4.3 illustrates the design and concept of passive monitoring chips, which allow extracting various circuit parameters from a single scattering parameter (S-parameter) measurement. Passive means that these samples do not contain Josephson junctions, which are the source of nonlinearity in TWPA samples. Otherwise, these samples have designs nominally identical to TWPAs and therefore allow monitoring linear properties without nonlinear effects. Other sources of nonlinearity, such as the power dependences of the dielectric loss and the kinetic inductance, were not observed at the power levels and temperatures used.

Three variants of passive monitoring samples were implemented, differing in the inductor only. The first has a meander inductor with inductance L_m similar to that of the rf-SQUIDs used in TWPAs. The second has a superconducting strip with an inductance much smaller than L_m , and the third has a meander inductor

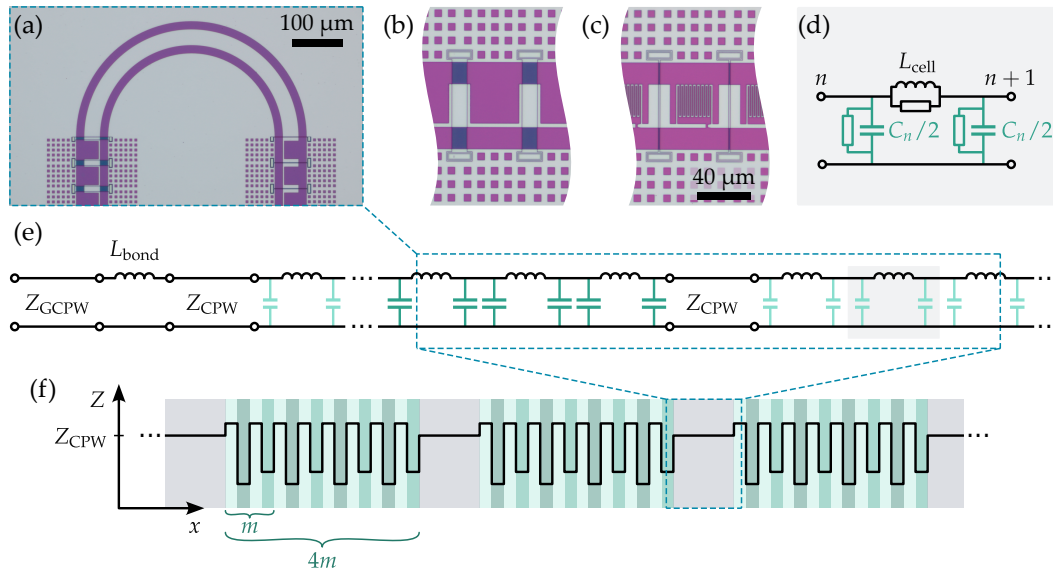


Fig. 4.3.: Passive mPCV-TWPA monitoring samples, nominally identical to TWPA samples but without Josephson junctions, to monitor linear properties without nonlinear effects. **(a)** The circuit consists of 13 LC ladder line arrays, connected by CPW segments, having $N = 1456$ unit cells in total. Monitoring samples of three inductor variants have been realized: the first identical to that of TWPA samples, the second and third are as shown in Panels **(b)**, **(c)**. Note that **(c)** does not contain Josephson junctions, so the SQUID loops are not closed. The capacitor design of all monitoring samples is identical to that of TWPA Sample A. **(d)** Circuit model of one unit cell, including resistors to account for dielectric loss and quasiparticle loss. **(e)** The circuit model of the whole circuit also contains bondwire inductances, and transmission lines modeling the CPWs on chip and the GCPWs on the PCB. **(f)** Illustration of the impedance variation along the line, not to scale. Periods m and $m/2$ cause the stopbands intended by mPCV, whereas period $4m$ causes super-period stopbands. Multi-reflections on the periodic structure, and CPW segments serving as reference impedance, allow determining inductance and capacitance values.

with an inductance much larger than L_m . Hence the three variants have average impedances smaller, close to, and larger than 50Ω .

Super-period stopbands

CPW segments appear at every $4m$ unit cells, connecting two adjacent arrays (Fig. 4.3a, f). This fourfold period $4m$ results in additional stopbands at quarters of the frequency of the stopband caused by period m of the multi-periodic capacitance variation. We refer to them as super-period stopbands. Their frequencies can be estimated as $\omega_{sp,i} \approx i\pi\omega_0/4m$, where index $i = 1, 2, 3, \dots$ indicates the order, whereas the mPCV stopbands occur at $\omega_{g,i} \approx i\pi\omega_0/m$ and at $2i\pi\omega_0/m$ due to periods m and $m/2$, respectively (Sec. 3.2.4). Super-period stopbands with $i = 4, 8, \dots$ coincide with the mPCV stopbands $\omega_{g,j}$ with index $j = 1, 2, \dots$. The actual frequencies at

which these stopbands occur deviate from these estimates due to dispersion. While being entirely unwanted in TWPAs, as described in detail in Sec. 4.5.2, super-period stopbands prove to be a valuable tool for circuit parameter extraction. A hand-waving explanation of how to extract circuit parameters using super-period stopbands goes as follows.

The super-period stopbands stem from reflections at interfaces between the LC ladder line, having local impedance $Z_n = \sqrt{L/C_n}$, and the CPW segments with impedance Z_{CPW} . In order to extract both L and C from a measurement, knowing the position of a stopband is not sufficient, because it depends on the product LC . The depth of the stopband, i.e., the insertion loss, depends however on an impedance mismatch involving the ratio L/C . The impedance mismatch also involves the impedance Z_{CPW} , which is however already known and serves as the reference impedance. For a characterization of the CPW see Appendix A.3. As a result, the position and the depth of the stopbands jointly provide L and C . Concretely, we employ the CTM model to fit scattering-parameter measurements, yielding several parameters besides L and C .

Calibrated scattering-parameter measurement in a cryogenic environment

The CTM model can be fitted to measured S-parameter spectra using least-squares optimization. For improved accuracy, not only the transmission S_{21} is measured, but the full scattering matrix, as it contains valuable information for a reliable fit. Such measurements are possible using the measurement setup shown in Fig. 4.4, featuring in-situ two-port microwave calibration in the cryogenic setup [154, 155, 156]. Commercially available SMA (SubMiniature version A) adapters of different lengths are used as Thru and Line standard, and SMA shorts as Reflect standards, enabling TRL (Thru-Reflect-Line) calibration. It is assumed that the geometries of the used standards do not change significantly when cooling down to 4.2 K. Adapters and coaxial cables between the switches are nominally identical. An uncertainty analysis of the presented calibration method is beyond the scope of this thesis. This TRL calibration works well in a frequency range of ca. 1–10 GHz, and gives reasonable results in a range of ca. 0.3–14 GHz, but has singularities below and above that range due to the lengths of the adapters used as Thru and Line standards (phase difference of the Thru and Line standards approaches 0° or 180°). For an increased frequency range, more than one Line standard should be used.

In the present work, this calibration method was used only for passive samples and devices. An extension suitable for sensitive active devices such as TWPAs, including input attenuation of ≥ 40 dB and output amplification is, however, possible [155]. All S-parameter measurements on TWPAs were performed using

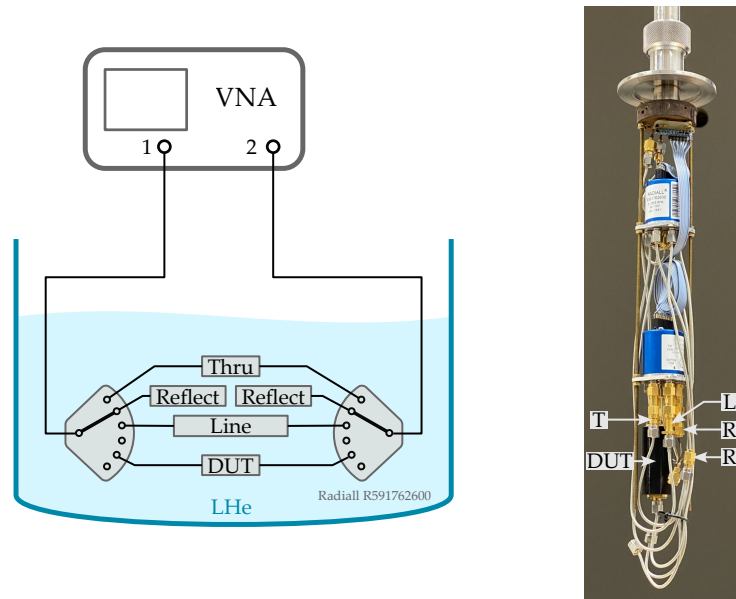


Fig. 4.4.: Setup for TRL-calibrated scattering parameter measurements in liquid helium. The calibration standards Thru (T), Reflect (R), Line (L), and the device under test (DUT), are selected in-situ via electromechanical switches.

a simpler Thru calibration, involving a separate cooldown in which the sample is replaced by a female-female SMA adapter serving as Thru standard. With this method, only S_{21} can be measured with reasonable calibration accuracy. A comparison of TRL and Thru calibrated measurements on the same sample is shown in Fig. 4.5.

Results

Figures 4.5–4.7 show TRL-calibrated S-parameter measurements of passive monitoring samples D, E, and F. The CTM model was fitted to $|S_{11}|$ and $|S_{21}|$ using least-squares optimization. Also shown is the dispersive part of the phase, where the linear part of the phase, $k_0 \propto \omega$, was subtracted from $k = -\arg(S_{21})/N$. Given that the fitting algorithm did not take into account the phase, the excellent agreement between measurement and model in both magnitude and phase highlights the accuracy and consistency of the CTM model. In particular, the position and magnitude of the super-period stopbands, as well as the sidelobes of the main stopbands, are well reproduced by the model. Deviations can be attributed to microwave background transmission, mostly via cavity modes in the sample package, and to significantly reduced accuracy of the TRL calibration for frequencies below 1 GHz and above 10 GHz, where the phase difference of the Thru and Line standards approaches 0° or 180° , respectively. In the fits, the values for ϵ_{CPW} and ϵ_{GCPW} were taken as fixed values, obtained from CPW resonator measurements

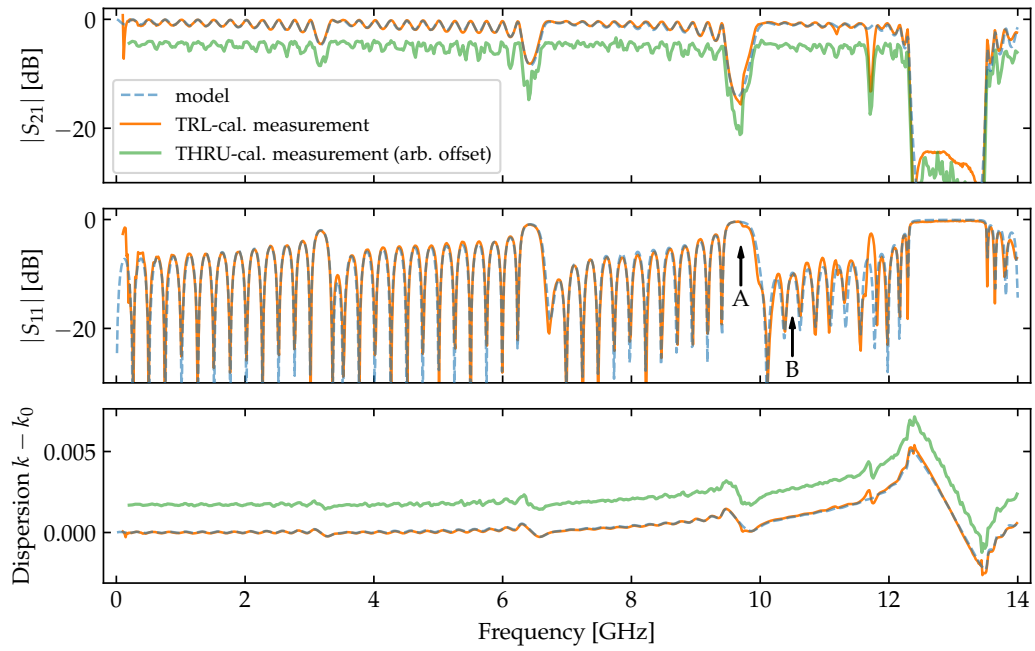


Fig. 4.5.: Fit of the cascaded-transfer-matrices model to measured S-parameter spectra of passive monitoring sample D. The three distinct dips (peaks) in $|S_{21}|$ ($|S_{11}|$) correspond to super-period stopbands, with the third one identified by letter A. The dips (peaks) between the super-period stopbands are sidelobes of the primary stopbands. One sidelobe is marked by B. The fit reproduces the TRL-calibrated measurement well and allows for the extraction of circuit parameters. The accuracy of the fit greatly benefits from using TRL calibration instead of a simpler Thru calibration. The green curves have an offset for visibility.

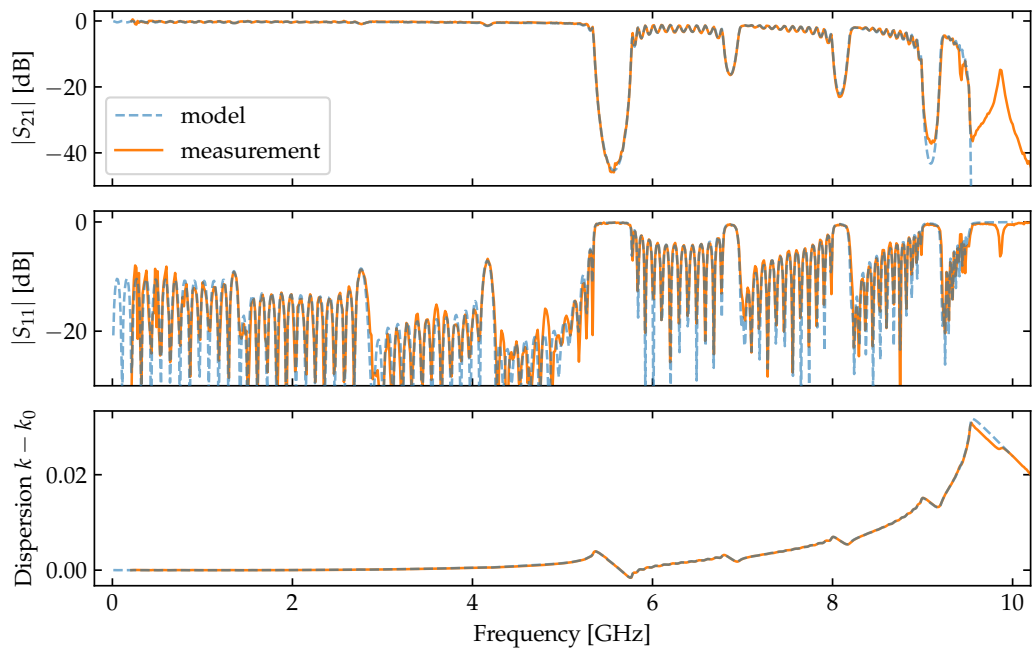


Fig. 4.6.: Fit of the CTM model to measured S-parameter spectra of passive monitoring Sample F. Stopbands occur at lower frequencies due to larger inductance. The $|S_{21}|$ peak at ca. 10 GHz stems from a cavity mode in the sample package.

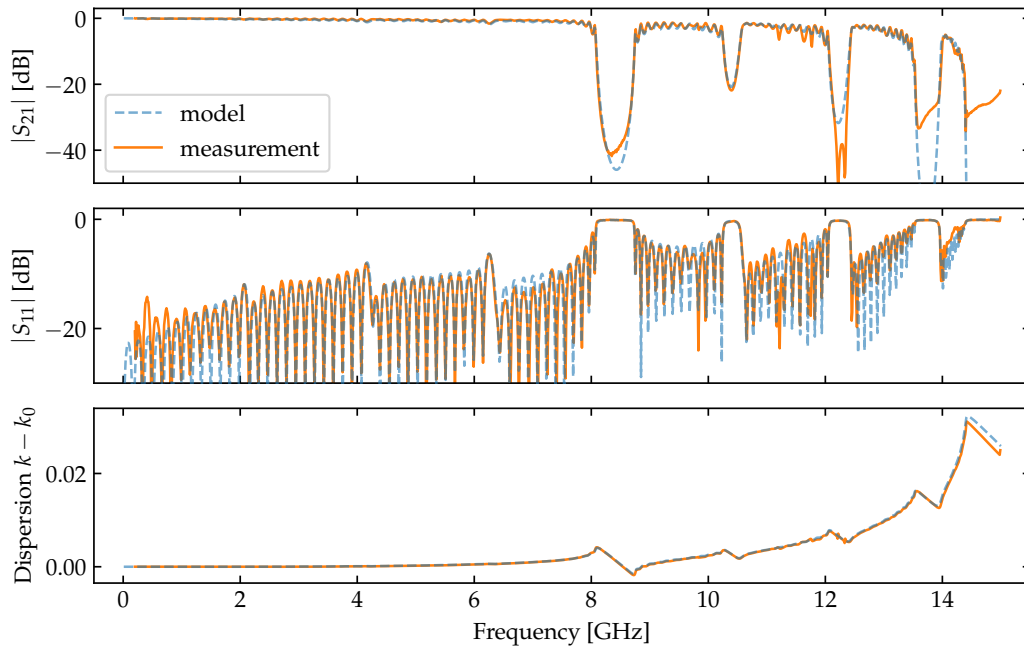


Fig. 4.7.: Fit of the cascaded-transfer-matrices model to measured S-parameter spectra of passive monitoring Sample E. The average impedance of Sample E is close to 50Ω . Apart from having only 1456 unit cells and a smaller variation depth ζ_1 , leading to a smaller first stopband, the sample is nominally identical to the TWPA shown in Fig. 4.8.

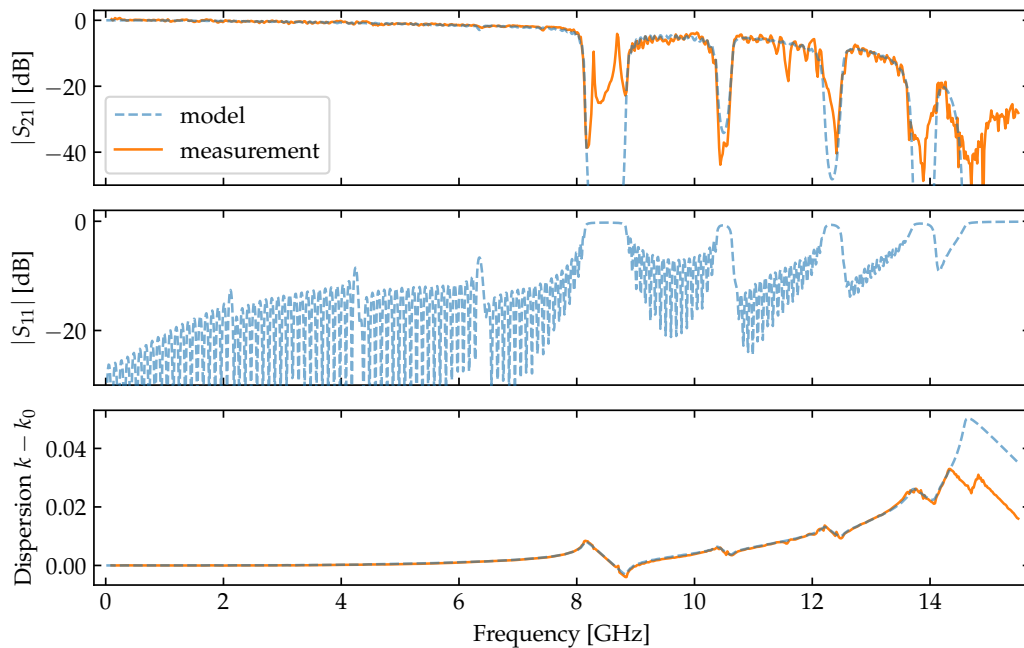


Fig. 4.8.: Fit of the CTM model to measured S_{21} spectrum of the TWPA. The spectrum is measured at a magnetic flux later used for three-wave-mixing parametric amplification. The setup used, shown in Fig. 4.11b, does not allow measurement of S_{11} . The curve for S_{11} is the reconstruction using the CTM model. It shows strong reflections, although the nominal impedance is close to 50Ω , which is a consequence of the sidelobes of the stopbands. Above 8 GHz the S_{21} spectrum is hampered by background transmission via cavity modes in the sample package.

(Appendix A.3) and GCPW simulations, respectively. The values of Z_{CPW} are close to the value of 50.22Ω obtained from the resonator measurement.

Tab. 4.2.: Fitting results of passive monitoring samples.

	Sample D	Sample E	Sample F	Sample G
L_{cell}	40.7 pH	94.1 pH	215.2 pH	41.2 pH
$\bar{C}$	42.6 fF	44.8 fF	45.6 fF	40.6 fF
ζ_1	0.107	0.105	0.108	0.104
ζ_2	0.676	0.644	0.620	0.662
$R_{\text{sg,eff}}$	100 k Ω	70 k Ω	20 k Ω	100 k Ω
$\tan \delta$	0.97×10^{-3}	1.07×10^{-3}	0.65×10^{-3}	0.95×10^{-3}
Z_{CPW}	49.9 Ω	49.7 Ω	50.4 Ω	50.5 Ω
ε_{CPW}	5.756	5.756	5.756	5.756
Z_{GCPW}	52.6 Ω	52.5 Ω	52.6 Ω	53.4 Ω
$\varepsilon_{\text{GCPW}}$	2.3	2.3	2.3	2.3
L_{bond}	270 pH	299 pH	239 pH	310 pH

Table 4.2 gives an overview of the fitting results of passive monitoring samples D–G. Sample G is of the same wafer as TWPA samples A–C, but the other passive chips from this wafer were not functional due to short circuits. Samples D–F are from another wafer. Samples D and G have identical designs. The quantity most important for further TWPA analysis is the average ground capacitance $\bar{C}$. All other quantities can be determined from CTM fits on TWPA small-signal transmission spectra. As discussed in Sec. 4.1, the average capacitance is ca. 30% larger than targeted, because stray capacitances were underestimated while designing the samples. Differences in $\bar{C}$ across Samples D–F are due to different cell inductance geometries, causing different stray capacitances. The inductor design of Sample E is identical to that of TWPA samples, so it is assumed that the stray capacitance per unit cell is the same for Samples A–C and E. Comparing Samples D and G, being from two different wafers, the $\bar{C}$ values differ by 2 fF, possibly due to slightly different PECVD-SiO_x thicknesses. The fitting uncertainty is ± 0.4 fF. The dominating source of uncertainty is the value of Z_{GCPW} , which is not well known. Repeating the CTM fits with Z_{GCPW} constrained between 50–55 Ω results in an uncertainty of ca. ± 2.8 fF for $\bar{C}$. Note that $\varepsilon_{\text{GCPW}}$ has only a small influence on the results. Therefore, we assume $\bar{C} = (42.8 \pm 3.2)$ fF for the TWPA samples A–C.

The values of the loss tangent $\tan \delta \approx 1 \times 10^{-3}$ is smaller than the typical literature value of 3×10^{-3} for PECVD-SiO_x [84], which is attributed to the fact that it was measured at 4.2 K, where two-level systems are expected to be saturated by thermal excitations. Josephson junctions are absent in these samples, so the effective subgap

resistance is entirely given by the surface resistance R_s , see Eq. 4.8. Comparing the R_s values at 10 GHz with an evaluation of Eq. A.56, the values differ by 20–55%, which is of the order of the fitting uncertainty of $R_{sg,eff}$, suggesting agreement. The bond wire inductances, which are on the order of 300 pH, are reasonable given the rule-of-thumb estimate of 1 nH/mm for a thin wire, and considering a length of ca. 500 μm , c.f. Fig. 4.1b.

4.2.4 Nonideal rf-SQUIDs

To determine the quantities of the rf-SQUID model of a TWPA unit cell (Fig. 4.2c), the small-signal transmission spectrum of the TWPA was measured at different flux-bias values, and the CTM model was fitted to the spectra. One exemplary spectrum and the corresponding CTM model are shown in Fig. 4.8, measured at the flux value later used for three-wave-mixing parametric amplification. Powers were small enough that the nonlinear effects reported in Sec. 4.3 were not observed.

A bias-current sweep of the spectrum is shown in Fig. 4.9. Here, the TWPA is current-biased by a DC current I_{dc} resulting in a flux $\Phi_{ext} = L_m I_{dc}$ in each SQUID loop. This bias modulates the SQUID inductances of the TWPA, and shifts the stopbands periodically with the period of one flux quantum. The origin of the lossy behavior close to half flux quanta is not yet understood, but is far from the flux value marked as a dotted line. Tracking the shift of the first stopband by individually fitting the CTM model to the S_{21} spectra at each I_{dc} value yields $L_{cell}(I_{dc})$ (circles in Fig. 4.9). For these fits, only the frequency range around the first stopband, 6–10 GHz, was considered, and parasitic cavity resonances (c.f. Fig. 4.8) were masked out. Further, the fit uses the fixed capacitance value $\bar{C} = (42.8 \pm 3.2)$ fF, determined from passive monitoring samples (Sec. 4.2.3), resulting in a proportional uncertainty of $\pm 7.5\%$ for $L_{cell}(I_{dc})$.

The system of Eqs. 4.2–4.5 describes the inductance of one unit cell, L_{cell} . Fitting these equations to the experimentally determined values of $L_{cell}(I_{dc})$ provides the circuit parameters of the nonideal-rf-SQUID model. Figure 4.9 displays the model (solid lines), which reproduces experimental data well. The nonlinearity coefficients β and γ , given by Eqs. 4.6 and 4.7, are also shown and indicate the operation conditions around the flux point for three-wave-mixing parametric amplification. The circuit parameters are summarized in Tab. 4.3. Remarkably, L_m and β_L are known with an accuracy independent of the uncertainty of the CTM fit, the latter being ca. 8% for L_{cell} , because they depend on the period and the shape of the interferometric bias-current-modulation pattern, respectively. Moreover, the value of the critical current I_c is very close to values obtained from DC measurements

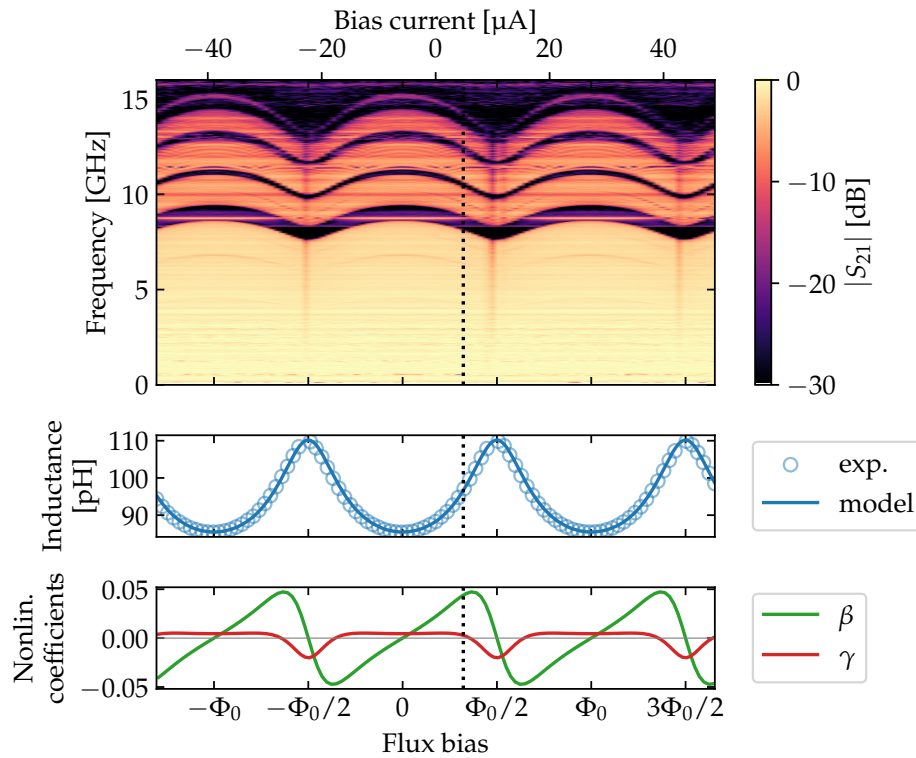


Fig. 4.9.: Bias current modulation. Top: Small-signal transmission, modulated by a bias current. Visible are the first and the second stopband and three super-period stopbands in between. Horizontal traces in the transmission spectrum are unwanted cavity modes in the sample package. Center: The inductance is extracted from fitting the stopband around 8–9 GHz to the CTM model (circles). The solid line depicts the nonideal-rf-SQUID model shown in Fig. 4.2c. Bottom: Second-order and third-order nonlinearity coefficients, β and γ , respectively, of the same model. The dotted vertical lines mark the flux later used for three-wave-mixing parametric amplification. At this flux value, β is close to its maximum and $\gamma \approx 0$. The spectra were measured with current bias in the setup presented in Fig 4.23 at ca. 20 mK.

on Josephson junction test structures (Appendix A.2.1), yielding $I_c = 0.99 \mu\text{A}$ for a junction area of $1 \mu\text{m}^2$.

The rf-SQUID has only a slight temperature dependence between 4.2 K and 0 K. The critical currents of both Josephson junctions increase when decreasing the temperature (see Appendix A.2.1), whereas the surface inductance, which is roughly 10% of L_m , decreases with decreasing temperature (see Appendix A.2.2). The latter leads to a decrease in L_m of ca. 0.3%. Accounting for both effects, β_L increases by ca. 5% from 4.2 K to 0 K.

Tab. 4.3.: Fitting results of the nonideal rf-SQUID model. Uncertainties are those given by the fitting algorithm.

Sample B	
L_m	$(62.5 \pm 0.3) \text{ pH}$
L_f	$(31.9 \pm 0.4) \text{ pH}$
$L_{p,g}$	$(6.7 \pm 3.5) \text{ pH}$
I_c	$(0.97 \pm 0.017) \text{ }\mu\text{A}$
$I_{c,L}/I_c$	14 (fixed)
$L_{J,L}$	$(24.3 \pm 0.4) \text{ pH}$
L_p	$(31.0 \pm 3.9) \text{ pH}$
$L_{\text{cell}}(\phi_{\text{dc}} = \pi/2)$	$(94.45 \pm 0.09) \text{ pH}$
L_{loop}	$(93.5 \pm 4.2) \text{ pH}$
α_p	(0.33 ± 0.03)
β_L	(0.275 ± 0.008)

4.2.5 Summary: Circuit model

Concluding this section, a detailed circuit model of the TWPA was developed. The model gives insight into a number of nonidealities of the device, and allows for gain simulation and deeper analysis of the device using WRspice circuit simulations as well as for further optimization of future TWPA designs (Sec. 4.5.3). Furthermore, the circuit model of the nonideal rf-SQUID enables input-power calibration (Sec. 4.3).

The circuit parameters of the model of the TWPA are summarized in Tab. 4.4. Most parameters were determined from small-signal microwave measurements on the TWPA itself, as described in the previous subsection. Parameters ζ_1 and ζ_2 are taken as mean values from CTM fits on flux-biased TWPA transmission spectra. Since the WRspice model used later in this work does not feature frequency-dependent resistors required to correctly model dielectric loss, $\tan \delta$ was set to zero, and all loss is approximated by a constant $R_{\text{sg,eff}}$, also obtained from the previous CTM fits. The values of $\bar{C}$ and L_{bond} were obtained by microwave measurements on passive monitoring samples (Sec. 4.2.3). For the value of C_J see Appendix A.2.1 and for the CPW values see Appendix A.3. Further complementing measurements on Josephson-junction test structures, including a discussion of tunnel-barrier quality and critical-current spread of the Josephson junctions, and measurements of the Nb-film characteristics are presented in Appendix A.2.

Tab. 4.4.: Circuit parameters of TWPA Sample B.

Sample B	
Meander loop inductance	$L_m = (62.5 \pm 0.3) \text{ pH}$
Parasitic loop inductance	$L_p = (31.0 \pm 3.9) \text{ pH}$
Passive inductance	$L_f = (31.9 \pm 0.4) \text{ pH}$
Critical current	$I_c = (0.97 \pm 0.02) \mu\text{A}$
Junction capacitance	$C_J = 45 \text{ fF}$
Effective subgap resistance	$R_{\text{sg,eff}} = (2.34 \pm 0.14) \text{ k}\Omega$
Average ground capacitance	$\overline{C} = (42.8 \pm 3.2) \text{ fF}$
Variation depth 1	$\zeta_1 = 0.128 \pm 0.002$
Variation depth 2	$\zeta_2 = 0.588 \pm 0.012$
CPW impedance	$Z_{\text{CPW}} = 50.22 \Omega$
CPW eff. permittivity	$\epsilon_{\text{CPW}} = 5.756$
GCPW impedance	$Z_{\text{CPW}} = (52.6 \pm 2.5) \Omega$
GCPW eff. permittivity	$\epsilon_{\text{GCPW}} = 2.3$
Bond wire inductance	$L_{\text{bond}} = (270 \pm 40) \text{ pH}$

4.3 rf-SQUID power dependence: Calibrating the input power

When powers of propagating waves in the TWPA are increased, the behavior of the rf-SQUIDs deviates from the small-signal case. The nonlinearity of the rf-SQUIDs causes Kerr effects like self-phase and cross-phase modulation. The former is visible in the S_{21} spectra shown in Fig. 4.10 as a power-dependent shift of the stopbands. The latter can be observed in two-tone spectroscopy (Fig. 4.22 in Sec. 4.5.5), where a strong pump and a weak probe tone are applied. This section discusses the power-dependent shift of the rf-SQUID inductance, quantitatively extracted from self-phase modulation in S_{21} spectra. The result is an input-power calibration, used for gain saturation measurements later (Sec. 4.5.4).

Nonlinear effects in the transmission spectrum

Depending on the sign of the nonlinearity coefficient γ of the rf-SQUID, referred to as the Kerr nonlinearity, the SQUID inductance increases or decreases, and stopbands shift to lower or higher frequencies (Fig. 4.10). For some flux-bias values the shift of the stopbands is only minimal, called the Kerr-free regime (Fig. 4.10b). The second-order nonlinearity (coefficient β) mediates unwanted effects like second-harmonic generation, which leads to power leakage to higher frequencies. This can be observed as a lossy region in Fig. 4.10b between -75 and -50 dBm and at frequencies of ca. 0–6 GHz, where phase-matching between the probe tone and its second harmonic is approximately achieved. When the second harmonic of the probe tone falls into one of the stopbands, its growth is strongly suppressed, and power leakage of the probe tone is reduced. This is the case for frequencies around the dotted line in Fig. 4.10b, where the second harmonic falls into the first stopband. For efficient three-wave-mixing parametric amplification, the pump is placed above the first stopband, such that its second harmonic falls into the second stopband. Some third-harmonic-generation loss can be observed in Fig. 4.10a.

Determining the rf-SQUID inductance from transmission spectra

The inductance of the rf-SQUID and its power dependence can be determined from the phase of S_{21} spectra. Additional to the phase shift caused by the Kerr nonlinearity, the phase has a strong shift caused by second-harmonic generation. To avoid the latter, the phase is evaluated at half of the frequency of the first stopband, ca. 4–4.5 GHz, such that the second harmonic is suppressed strongly and does not

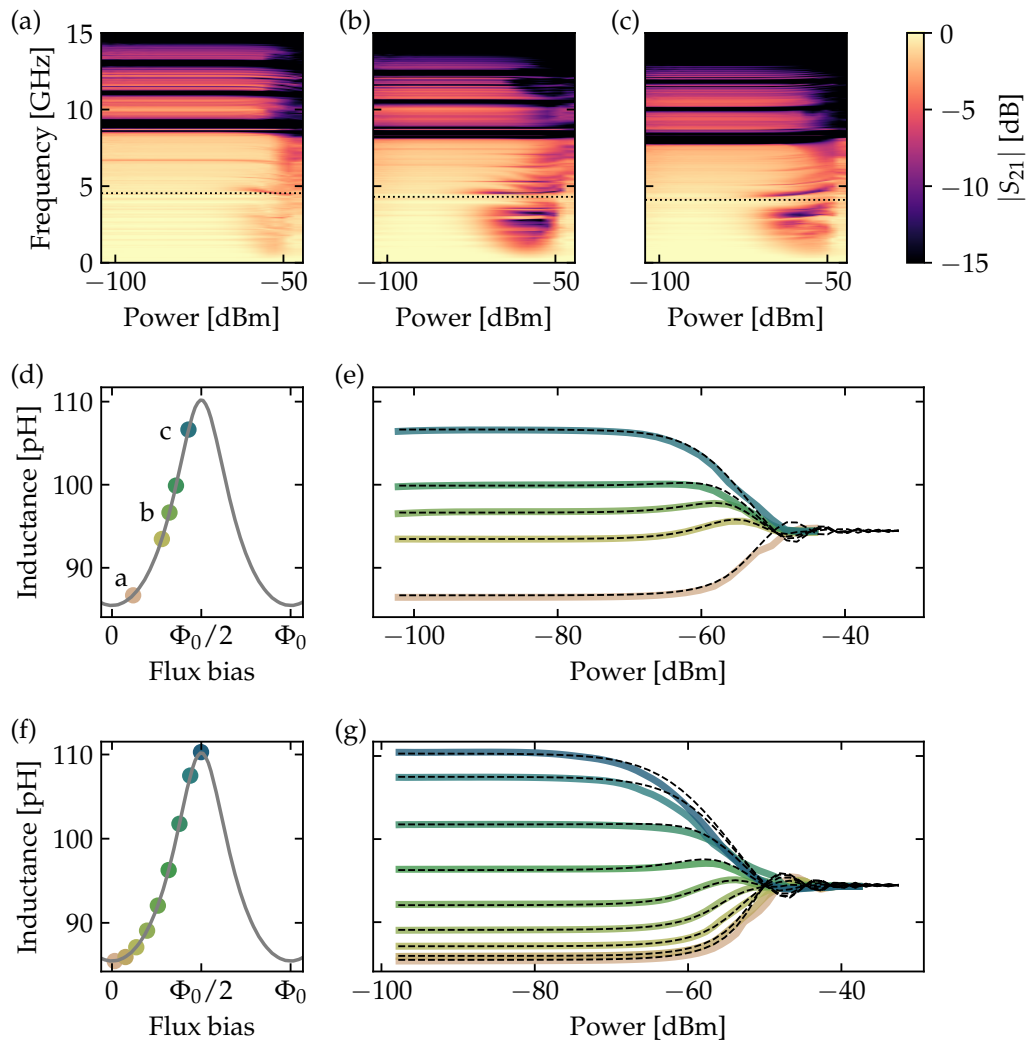


Fig. 4.10.: Power dependence of transmission and SQUID inductance. **(a)–(c)** Power dependence of $|S_{21}|$ at different flux-bias values as annotated in **(d)**. The stopbands shift notably for larger powers due to self-phase modulation of the probe tone. The direction of the shift depends on the Kerr nonlinearity, which is, from left to right, positive, vanishing, and negative, respectively. The lossy region most dominant in **(b)** stems from second-harmonic generation due to quadratic nonlinearity. For frequencies where the second harmonic falls into the first stopband, second-harmonic generation is inefficient (around the dotted lines). **(d), (e)** Flux and power dependence of the SQUID inductance at 4.2 K, measured with magnetic-field bias. Colored dots are extracted from S_{21} spectra using a CTM fit at low powers (−102 dBm). Solid lines with corresponding colors depict the power dependence of the inductance and are extracted from $\arg S_{21}$ at frequencies where S_{21} is not hampered by second-harmonic generation, marked by dotted lines in **(a)–(c)**. Dashed lines are a WRspice simulation of the nonideal-rf-SQUID model, showing good agreement. The power scale of the measurement was aligned to that of the simulation, which serves as an input-power calibration. For the lines in the center, the Kerr shift is very small. The optimal magnetic field, used for three-wave-mixing parametric amplification, is that marked by letter b. At very high amplitudes the SQUID behaves as a linear inductor independent of the flux bias, with a value approximately equal to that of passive monitoring sample F. The spectra for panels **(d), (e)** were measured with the setup shown in Fig. 4.11b. **(f), (g)** Flux and power dependence of the SQUID inductance at ca. 20 mK, measured with current bias in the setup presented in Fig 4.23.

hamper the phase of the probe tone. The dimensionless wavenumber relates to the phase of S_{21} as

$$k = -\frac{\arg(S_{21})}{N}, \quad (4.21)$$

with the number of unit cells N . When the power is increased, the wavenumber changes by

$$\delta k = -\frac{\arg(S_{21}(P)) - \arg(S_{21}(P_0))}{N}, \quad (4.22)$$

where P_0 denotes the lowest power of the measurement. At this power the CTM model is fitted to S_{21} to extract the inductance $L_{\text{cell}}(P_0)$, referred to as the small-signal inductance. At frequencies far below the first gap the chromatic dispersion of the TWPA is small, so k is approximately linear and can be expressed via the phase velocity v_{ph} as

$$k_{\text{lin}}(P) = \frac{a\omega}{v_{\text{ph}}} = \omega \sqrt{L_{\text{cell}}(P)\bar{C}}. \quad (4.23)$$

Rearranging Eq. 4.23 and inserting $k_{\text{lin}}(P) = k_{\text{lin}}(P_0) + \delta k$ then yields

$$L_{\text{cell}}(P) = \frac{1}{\bar{C}} \left(\frac{k_{\text{lin}}(P_0) + \delta k(P)}{\omega} \right)^2. \quad (4.24)$$

Phase offsets in the measurement, stemming e.g. from chromatic dispersion, drop out and have no influence. The method presented here is more precise than fitting the CTM model to the S_{21} spectra, because the behavior of the stopbands changes drastically with increasing power for non-vanishing Kerr nonlinearities [50]. Therefore, the CTM model is not valid for larger powers.

Figures 4.10e,g show the experimentally determined power dependence of L_{cell} in comparison to that of a WRspice simulation of a single nonideal rf-SQUID following the model presented in Sec. 4.2.4 (simulation model refer to Fig 3.1c and inductance extraction see Eq. 3.11). The power scale of the experimentally determined curves was aligned to that of the simulation. Simulation and experiment are in good agreement. Deviations of the curves occur predominantly for curves with large power dependence, when the second harmonic of the frequency, at which $\arg(S_{21})$ is evaluated, does not fall into the stopband anymore due to the large shift of the latter. It can be seen that the inductance of the green curve, marked by letter b in Fig. 4.10d, has the lowest power dependence until ca. -55 dBm. This is the optimal flux bias for three-wave-mixing parametric amplification, because four-wave-mixing and related effects such as self-phase and cross-phase modulation are vanishingly small and the second-order nonlinearity is large.

Behaviour of rf-SQUIDs at high current amplitudes

A remarkable property of rf-SQUIDs is that they do not possess a maximum current after which the device switches to the voltage state, until the current approaches the critical current of the superconducting film. This advantage of rf-SQUID-based TWPAs contrasts TWPAs based on SNAILs, dc-SQUIDs, or single Josephson junctions as nonlinear inductive elements, which possess such a maximum current, placing a limit on the usable pump current. Instead, at very high amplitudes the rf-SQUID behaves as a linear inductor independent of the flux bias and with vanishing nonlinear behavior. The high-power inductance value is approximately equal to that of passive monitoring sample F, showing that the high-power limit of the rf-SQUID inductance is $L_m + L_f$, as expected.

Input power calibration

Aligning the simulated and experimental curves can be used to calibrate the input-power level at the sample, important to determine the saturation power of the TWPA (Sec. 4.5.4). The frequency range of 4–4.5 GHz, for which this calibration is valid, coincides with the center of the signal band. This method has several uncertainties. An important contribution is the insertion loss of the sample, which is ca. 1 dB at 4–4.5 GHz for all flux bias values. The lossless simulation is aligned to the experimentally determined average inductance of all unit cells of the TWPA. Accounting for insertion loss, the input power is higher than the average power inside the sample. Hence, 0.5 dB were added. Since the exact power distribution inside the TWPA is not known, we take 0.5 dB as uncertainty. Another source of uncertainty is the microwave measurement itself. Accounting for frequency dependent insertion losses and return losses of infrared filters, coaxial cables, etc., in the range 4–4.5 GHz, and for power level uncertainties of the VNA, we estimate 1 dB of uncertainty. Individually aligning the curves via least-squares optimization results in an fitting uncertainty of 0.6 dB. Altogether, the alignment uncertainty is 2.1 dB.

The strength of the method is the confidence in the SQUID model. Although the cell inductance, determined via a fit of the CTM model to S_{21} spectra, has an uncertainty of ca. 8%, the parameters L_m and β_L are known much more precisely. This is because they are determined interferometrically via flux modulation and depend on the period and the shape of the flux modulation pattern, respectively, independent of the CTM fit uncertainty. Using 8% lower L_{cell} values in the whole input power calibration procedure, the power alignment is different only by 0.05 dB. Apart from that, an advantage of the presented method is that it utilizes the amplifier under test itself to calibrate the input power, rather than relying on external devices, such

as qubits [49, 90]. This simplifies the measurement setup, and it enables power calibration at temperatures like 4.2 K, far above suitable operating conditions for qubits.

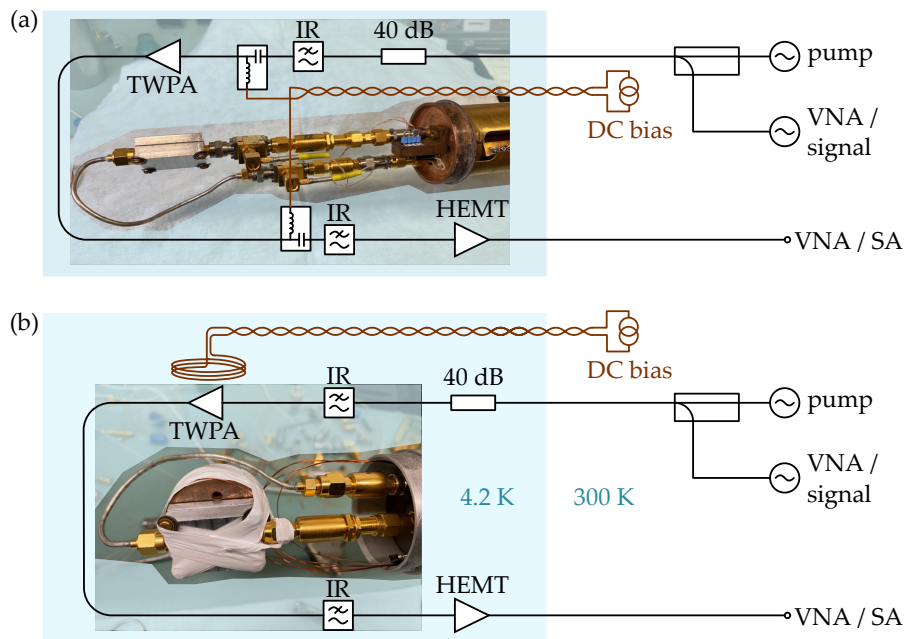


Fig. 4.11.: Setup for precharacterization in liquid helium. The setup has two variants: one with bias tees for current biasing and one with a solenoid for magnetic-flux biasing. Infrared (IR) filters are coaxial Eccosorb filters. Not shown is a Cryoperm® cup for magnetic shielding. The VNA is calibrated by replacing the TWPA by an female-female SMA adapter, serving as a Thru standard. A calibration requires a dedicated cooldown.

4.4 Parametric deamplification by frequency conversion

In TWPA development at PTB, many TWPA samples suffered from insertion loss beyond what could be explained by dielectric loss or quasiparticle loss. This behavior was also observed by other groups working on 3WM TPWAs. This section discusses the origin of this hidden loss mechanism and methods to mitigate it.

To assure that the origin of the large insertion loss is not dielectric loss or surface-resistance loss, we measured (active) TWPA samples along with cofabricated passive monitoring samples, where the only difference is the absence of Josephson junctions (see Sec. 4.2.3). The small-signal transmission of the passive samples showed small insertion losses attributable to a loss tangent on the order of 10^{-3} at 4.2 K, as expected for the used dielectric (c.f. Tab. 4.2). The transmission of many TWPA samples, in contrast, showed much higher insertion losses, as shown in the following.

This lossy phenomenon exhibits two distinct fingerprints. First, it occurs in a quasi-frequency-periodic manner (Fig. 4.12a,b), where high insertion loss coincides with a small period. This pattern resembles the experimentally measured output power of the sum frequency of signal and pump $f_{p+s} = f_p + f_s$ of a kinetic-inductance-based 3WM TWPA, recently reported by Klimovich et al. [157]. Second, the lossy features modulate with the flux bias of the rf-SQUIDs, precisely such that the highest loss occurs where the second-order nonlinearity coefficient β is large (Fig. 4.12c–e). The loss appears as a linear effect, remaining unaltered for a large range of probe tone powers. When the power is further increased, or a strong pump is applied, the loss reduces gradually. This presents a caveat when measuring parametric signal gain using a pump-on-pump-off calibration [75, 158, 53] instead of a Thru-calibration. Because the former does not account for this insertion loss, it only measures the relative increase in signal amplitude instead of the absolute signal gain. Therefore, this method does not distinguish between decreased signal loss and true signal gain.

Spectroscopy of frequency-conversion loss

The correspondence between high loss and high quadratic nonlinearity suggests parametric frequency conversion as the loss mechanism. This three-wave-mixing effect is driven by a strong wave at a low frequency f_d , which generates an upper and a lower sideband frequency of a weak probe tone at frequency f_s , as illustrated in Fig. 4.13. These first-order sidebands have frequencies $f_u = f_s + f_d$ and $f_l = f_s - f_d$, respectively. Higher-order processes and harmonics of f_d also create higher-order sidebands at frequencies $f_s \pm if_d$, where $i = 2, 3, 4, \dots$. Through sideband

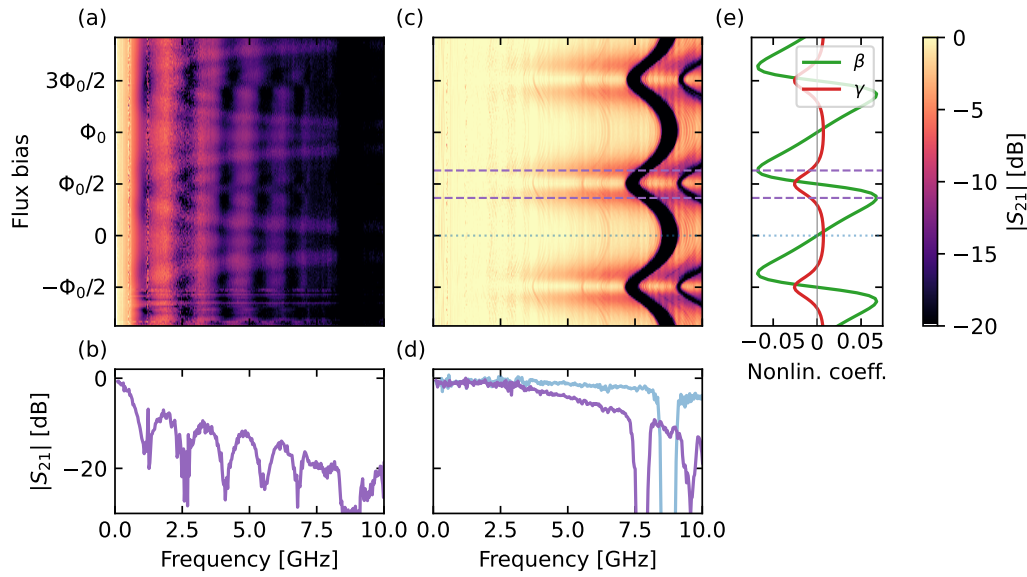


Fig. 4.12.: Symptoms of frequency-conversion loss due to noise interference in the small-signal transmission spectrum of a TWPA. **(a), (b)** Unfiltered noisy current biasing causes frequency-periodic transmission dips. **(c), (d)** Low-pass filtering of the DC current bias reduces this loss. At bias values leading to high quadratic nonlinearity coefficient β [Panel **(e)**], the TWPA exhibits significantly greater loss (dashed lines) compared to zero quadratic nonlinearity (dotted line). The linecuts in **(b)** and **(d)** are at flux bias values of high (purple) or zero (blue) quadratic nonlinearity. Both measurements were performed under similar conditions in the same cooldown, differing only in the presence of a low-pass filter at room temperature used in **(c)** and **(d)**. The TWPA Sample H is similar to Sample A but from a previous generation.

generation, the transfer of power from the probe tone to the sidebands depletes the probe tone, resulting in frequency-conversion loss.

The strong drive tone f_d is not intentionally inserted into the system, but rather stems from the noise spectrum of galvanically connected electronic devices, or is picked up electromagnetically in the measurement apparatus. Figure 4.13 shows the output spectrum of TWPA Sample C with a fixed probe tone at 6 GHz, measured in the setup shown in Fig. 4.11a with a spectrum analyzer. The spectrum of the probe tone exhibits a series of sidebands of low spectral purity. The frequency spacing reveals a drive tone at 133 kHz, which is a typical clocking frequency of switching-mode power supplies, used in PCs, fluorescent lamps, and so on. The bandwidth of the first-order sidebands is 7 kHz, which is 70 times larger than the resolution bandwidth of the spectrum analyzer, corresponding to 18 dB. Therefore, the power contained in each sideband is only a few dB below that of the spectrally much purer probe tone. We observed a large number of sidebands, far beyond the six sidebands shown, in total causing frequency-conversion loss exceeding 10 dB

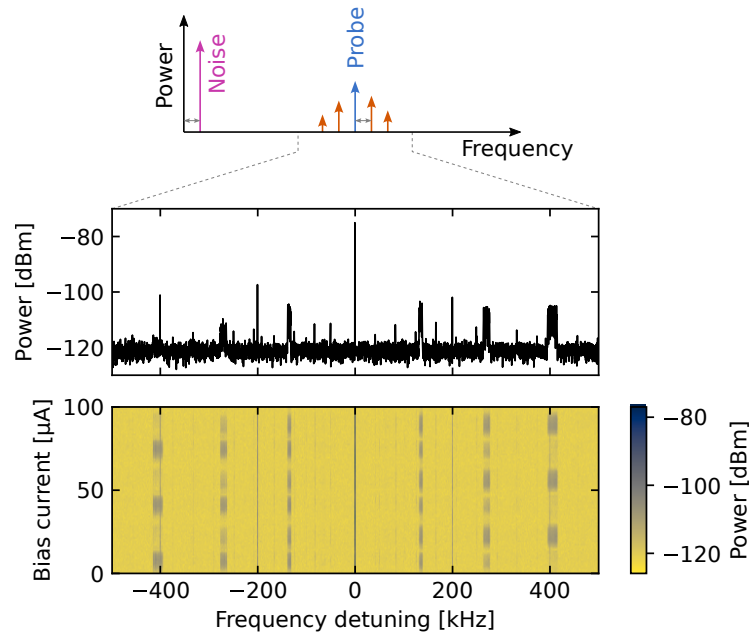


Fig. 4.13.: Sideband generation through frequency conversion. A picked-up noise tone generates sum-frequency and difference-frequency sidebands of a weak probe tone, depleting its power. In the measured output spectrum of a TWPA, the present sidebands reveal a noise tone at 133 kHz. Sideband amplitudes are modulated by the flux bias. Additional narrowband sidebands with a detuning of 200 kHz do not modulate with the flux bias and are attributed to components after the TWPA. The frequency of the probe tone is 6 GHz and the resolution bandwidth of the spectrum analyzer is 100 Hz.

for some bias current values. The bias current modulates the amplitudes of these sidebands, which agrees with the suspected 3WM process, being pronounced when the quadratic nonlinearity is large.

Analytical description of frequency conversion

To analytically describe frequency-conversion loss, for now we neglect higher-order sidebands and consider only three modes: the strong drive at frequency f_d , the weak probe tone $f_s > f_d$, and its first-order upper sideband $f_u = f_s + f_d$. For simplicity, we neglect the phase mismatch $\Delta k = k_u - k_d - k_s$, where k_u, k_d, k_s denote the corresponding wavenumbers. This is valid for frequencies well below

the first stopband, where $k \propto f$. The spatial evolution of their complex amplitudes $A_d(x), A_s(x), A_u(x)$ is given by the coupled-mode equations (see Sec. 2.4),

$$\frac{dA_u}{dx} = \frac{\beta}{2} k_d k_s A_d A_s = \kappa_u A_d A_s, \quad (4.25)$$

$$\frac{dA_d}{dx} = -\frac{\beta}{2} k_u k_s A_u A_s^* = -\kappa_d A_u A_s^*, \quad (4.26)$$

$$\frac{dA_s}{dx} = -\frac{\beta}{2} k_u k_d A_u A_d^* = -\kappa_s A_u A_d^*, \quad (4.27)$$

introducing the nonlinear coupling coefficients $\kappa_u, \kappa_d, \kappa_s$. Assuming strong and undepleted drive amplitude $A_d = \text{const.}$, and differentiating Eq. 4.25, we find

$$\frac{d^2 A_u}{dx^2} = \kappa_u A_d \frac{dA_s}{dx} \quad (4.28)$$

$$= -\kappa_u \kappa_s A_u |A_d|^2 \quad (4.29)$$

$$= -\delta^2 A_u, \quad (4.30)$$

with $\delta = \sqrt{\kappa_s \kappa_u} |A_d|$. Here, we treat the transmission line with multiperiodic capacitance variation as a spatially uniform continuum, having wavenumber $k \neq k(x)$, valid for small frequencies well below the first stopband. A solution of Eq. 4.30 is

$$A_u = A_{u,1} \sin \delta x + A_{u,2} \cos \delta x. \quad (4.31)$$

The boundary conditions at the TWPA input, $A_s(x=0) = A_{s0}$ and $A_u(x=0) = 0$, require $A_{u,1} = \kappa_u A_d A_{s0} / \delta = \sqrt{\kappa_u / \kappa_s} A_{s0}$ and $A_{u,2} = 0$, respectively. Differentiating Eq. 4.31 and rearranging Eq. 4.25, we find the spatial evolution of the probe tone and the sideband,

$$A_s(x) = A_{s0} \cos \delta x, \quad (4.32)$$

$$A_u(x) = \sqrt{\frac{k_s}{k_u}} A_{s0} \sin \delta x, \quad (4.33)$$

$$\delta = \frac{\beta}{2} k_d |A_d| \sqrt{k_s k_u}. \quad (4.34)$$

The coefficient $\sqrt{k_s / k_u} \approx \sqrt{f_s / f_u}$ in Eq. 4.33 reflects the different photon energies involved in the parametric process. Converting the node-flux amplitude $|A_d|$ to phase perturbation $\phi_d = k_d |A_d|$, and using $I_d = \varphi_0 \phi_d / L_{\text{cell}}$ (see Eq. 3.58), links δ to

the power of the drive tone, $P_d = I_d^2 Z_0 / 2$. Finally, converting Eqs. 4.32–4.33 in the same way, using the input power of the probe tone P_{s0} , yields

$$P_s(x) = P_{s0} \cos^2 \delta x, \quad (4.35)$$

$$P_u(x) = P_{s0} \sin^2 \delta x, \quad (4.36)$$

$$\delta = \frac{\beta L_{\text{cell}}}{2\varphi_0} \sqrt{\frac{2P_d k_s k_u}{Z_0}}. \quad (4.37)$$

This result shows a complete power transfer from the probe tone to the sideband at $x = i\pi/\delta$, where $i = 1, 3, 5, \dots$, and vice versa. This explains the frequency-periodic behavior of the frequency-conversion loss shown in Fig. 4.12a,b. Since $\delta \propto \beta$, it also explains the observation that when β is strong, frequency-conversion loss is pronounced (Fig. 4.12c,d). Of particular interest is the case for small drive frequencies $f_d \ll f_s$ (Fig. 4.13), for which $\delta \propto \sqrt{k_s k_u} \propto \sqrt{f_s(f_s + f_d)} \approx f_s$. Strikingly, the spatial frequency-conversion period π/δ depends on the drive power P_d , whereas for $f_d \ll f_s$, the entire frequency-conversion process does not depend on the drive frequency f_d .

Circuit simulation

To include now higher-order sidebands, which were observed in large numbers (Fig. 4.13), circuit simulations were performed in WRspice, applying a strong low-frequency drive tone and a weak probe tone. The circuit model was extracted from the data shown in Fig. 4.12c as explained in Sec. 4.2, and parameters are compiled in Tab. 4.5. Since frequency-conversion does not depend on the frequency of the noise tone $f_d \ll f_s$, it can be chosen freely in the simulation, and was set to 20 MHz to match the FFT frequency resolution. Simulating with lower frequencies f_d requires proportionally longer time series, which increases computational cost.

Figure 4.14 shows the simulated transmission spectrum in comparison to an experimental spectrum. The distinct quasi-frequency-periodic transmission dips originate in the sine-cosine-shaped power transfer between probe tone and sidebands, as shown in Fig. 4.14c for simulation and theoretical prediction, Eqs. 4.35–4.37. In case A, some power of the probe wave is lost to sidebands, leading to decreased transmission. For higher probe frequencies, the spatial frequency-conversion period $\pi/\delta \propto 1/f_s$ is smaller, and the power transfer reverses. In case B, power transfer reverses multiple times, and the nearly complete power transfer from probe tone to sidebands at the output node of the TWPA results in a transmission dip.

The theory prediction deviates notably from the simulation, which can be attributed to the shortcoming of including only one sideband in the theory. In

Tab. 4.5.: Circuit parameters of TWPA Sample H. Not mentioned parameters are identical to those of Tab. 4.4.

Sample H	
Meander loop inductance	$L_m = 68 \text{ pH}$
Parasitic loop inductance	$L_p = 10 \text{ pH}$
Passive inductance	$L_f = 38 \text{ pH}$
Critical current	$I_c = 1.3 \text{ }\mu\text{A}$
Eff. subgap resistance	$R_{\text{sg,eff}} = 2.2 \text{ k}\Omega$
Average ground capacitance	$\bar{C} = 42 \text{ fF}$
Variation depth 1	$\zeta_1 = 0.105$
Variation depth 2	$\zeta_2 = 0.54$
Number of cells	$N = 1904$

particular, the simulation does not show full recovery of the probe-tone power after power-transfer reversal, and also the periodicity in frequency is not consistent. Frequency-conversion occurs as a cascaded process generating a whole myriad of sidebands, all having different characteristic lengths after which power transfer reverses. The longer the tones interact, the broader this cascade of frequency-conversion processes becomes, and the more power diffuses to sidebands of higher and higher orders. This leads to the observation that each successive peak in the transmission spectrum exhibits a lower transmission value than the one before it.

Effect of filtering on frequency-conversion loss

Lowering the amplitude of the unwanted drive tone leads to transmission behavior as shown in Fig. 4.15. The measurement was performed in the same cooldown and on the same sample as the measurement presented in Fig. 4.12a, differing only in the presence of a low-pass filter at room temperature in the DC biasing lines. The filter is a second-order RC low-pass filter with a cutoff frequency of ca. 10 kHz, constructed symmetrically to suppress both differential-mode and common-mode noise. The setup used for both measurements also contained 1.5 m long twisted Thermocoax® lines between the 300 K flange and the mixing chamber plate ($\sim 20 \text{ mK}$) as a cryogenic low-pass filter on the DC lines (see Fig. 4.23). Thermocoax® coaxial lines, originally designed as flexible heating filaments, consist of a resistive center conductor, a MgO dielectric, and a stainless steel outer conductor. They exhibit low-pass characteristics at microwave frequencies with approximately 80 dB/m attenuation at 1 GHz and exceeding 120 dB/m between 2 and 40 GHz (measurement limited by the dynamic range of the VNA). The low-pass characteristic is expected to continue into the THz range, reaching more than

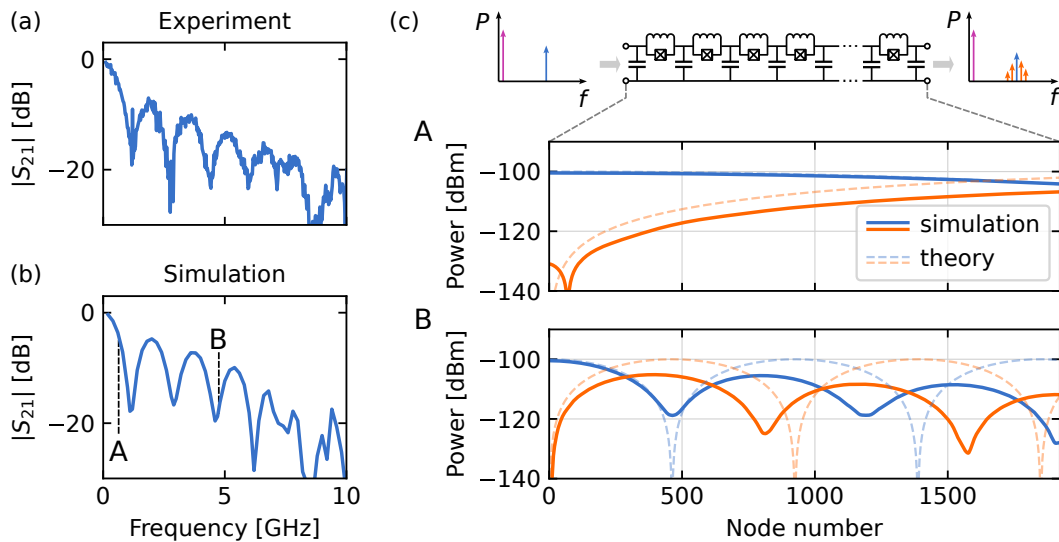


Fig. 4.14.: Origin of the quasi-frequency-periodic lossy transmission dips. **(a)** Small-signal transmission spectrum from the unfiltered measurement presented in Fig. 4.12a. **(b)** WRspice simulation with a 20 MHz noise tone. Its amplitude of $5.3 \mu\text{A}$ was chosen to visually fit the experimental data, and bias current $I_{\text{dc}} = 12 \mu\text{A}$ to achieve high β . **(c)** Spatial evolution of the probe-tone wave (blue) and the first-sideband wave (orange) at frequencies marked in Panel (b). Power is transferred back and forth between the waves as they propagate, leading to frequency-periodic features in the transmission. For the theory plots, $\pi/\delta = 13,273$ in case A, and 924 in case B to resemble the first complete power transfer at node 462.

130 dB/m at 1 THz [159]. However, at 10 MHz, the attenuation is only ca. 10 dB/m, not sufficient to filter out the apparent low-frequency (~ 100 kHz) noise causing frequency-conversion loss.

The simulation shown in Fig. 4.15 uses the same circuit model as in Fig. 4.14 but with a smaller amplitude of the drive tone. A higher drive frequency of 100 MHz was chosen to allow shorter simulation times. The remarkable quantitative agreement between experiment and simulation shows that frequency-conversion is with good certainty the dominant loss mechanism for the observed behavior.

Further reduction of frequency-conversion loss is observed in the transmission measurement on Sample B shown earlier in this work in Fig. 4.9 without additional low-pass filters at room temperature. This measurement was performed in the setup as shown in Fig. 4.23. The two main differences between the two setups used are a voltage source with lower noise (National Instruments NI9269 instead of Agilent B2901) and the presence of the directional coupler to couple probe tone and pump to the TWPA. The coupled port of a directional coupler exhibits high-pass characteristics, with a cutoff frequency of approximately 1.5 GHz for the device used. It is effectively intransparent to sub-GHz frequencies, thereby filtering out low-frequency noise in the microwave input line.

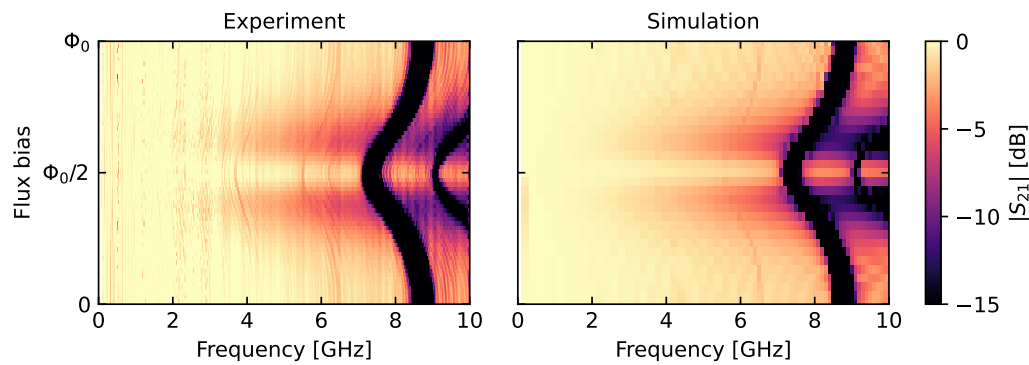


Fig. 4.15.: Flux-bias dependence of frequency-conversion loss. Left: Small-signal transmission spectrum taken from the filtered measurement presented in Fig. 4.12b. Right: WRspice simulation with a 100 MHz noise tone. Its amplitude of $0.3 \mu\text{A}$ was chosen to visually fit the experimental data. The probe amplitude was $0.03 \mu\text{A}$.

Interestingly, at flux-bias values of $0, \Phi_0/2, \Phi_0$, and so on, where the second-order nonlinearity coefficient β vanishes and the third-order nonlinearity coefficient γ is large, there is no notable frequency-conversion loss (Fig. 4.15). This highlights that frequency-conversion loss is a typical issue with 3WM TWPAs, but not with 4WM TWPAs. The reason for this behavior is the nonlinear phase-mismatch in 4WM TWPAs caused by self-phase and cross-phase modulation between the interfering tone and the probe tone, suppressing frequency-conversion. Another practical reason is that most 4WM TWPAs do not require biasing, thereby having only two (typically heavily filtered and attenuated) input and output microwave lines connected to the TWPA, and no DC connections that might introduce noise.

Conclusions

Concluding this section, parametric frequency conversion has been identified as the origin of the longstanding issue of large insertion loss in 3WM TWPA samples above what could be explained by dielectric or quasiparticle loss. The loss mechanism is driven by a noise tone of sufficient amplitude and at frequencies lower than the probe tone in a transmission measurement. This noise tone can be either inserted into the sample galvanically from noisy electronic devices, such as current sources, or it can be picked up electromagnetically, for example in ground loops or non-twisted DC lines. Therefore, measures to prevent frequency-conversion loss include using low-noise current sources in combination with low-pass filtering on the DC lines, twisted-pair DC wiring to prevent picked-up magnetic interference, and high-pass filters or a directional coupler on the AC lines.

While in this section frequency conversion was presented as an unwanted effect, it can be utilized intentionally for engineered deamplification. This approach is suitable for TWPAs with parametric amplification for forward-propagating signals and isolation through frequency conversion for backward-traveling signals, which is an area of ongoing research in wideband amplifier development.

4.5 Parametric amplification

This section describes the characterization of the amplification performance of the TWPA, comprising the signal power gain, the bandwidth, and the saturation power. The noise performance is described in the next section. While the preceding sections describe single-tone spectroscopy, i.e., probing the transmission of the TWPA with a single tone applied, this section is dedicated to two-tone spectroscopy, where a strong pump and a weak signal are applied to the TWPA input.

Due to the flux-tunability of the nonlinearity of the rf-SQUID based TWPA, the device features parametric amplification in both three-wave-mixing (3WM) and four-wave-mixing (4WM) operation, although it is designed and optimized for the former. The section starts with an overview of a multitude of parametric processes occurring at the optimal flux for 3WM parametric amplification, at which the Kerr nonlinearity vanishes. This is followed by the amplifier performance characterization under phase-matched 3WM conditions, the core part of this section. Finally, 4WM parametric amplification in two different regimes is shown.

4.5.1 A multitude of parametric processes

A nonlinear medium such as a TWPA generally facilitates a multitude of parametric processes. Tuning the magnetic flux (or the bias current) such that the Kerr nonlinearity vanishes, eliminates the possibility of 4WM processes. The major 3WM processes include downconversion, with frequency relation $f_i = f_p - f_s$, signal upconversion, $f_{p+s} = f_s + f_p$, and idler upconversion, $f_{p+i} = f_i + f_p$. The former two processes are of first order, while the latter is of second order, since it involves a product of a first-order process, i.e., the idler f_i . Higher-order processes occur, too, but are neglected in this discussion. Another 3WM process is second-harmonic generation of the pump, with frequency relation $f_{2p} = 2f_p$. Due to the small amplitudes of signal and idler, their harmonic generations are negligible. All these processes occur simultaneously in the TWPA. Their effect on the signal gain is illustrated by Fig. 4.16, where the signal gain spectrum is shown for a range of pump frequencies. The spectra were measured in the setup shown in Fig. 4.11b at 4.2 K. The flux bias is such that the Kerr nonlinearity vanishes. The pump power at the TWPA input is ca. -57 dBm. Red regions depict signal gain, while lossy regions are shown in greyscales. Super-period stopbands lead to distinct diagonal traces, an effect discussed in Sec. 4.5.2.

Whether a parametric process is enhanced or suppressed, depends on its phase-matching condition. A process is effectively phase-matched, if the phase mismatch Δk (for example $\Delta k = k_p - k_i - k_s$ for the downconversion process) leads to a long

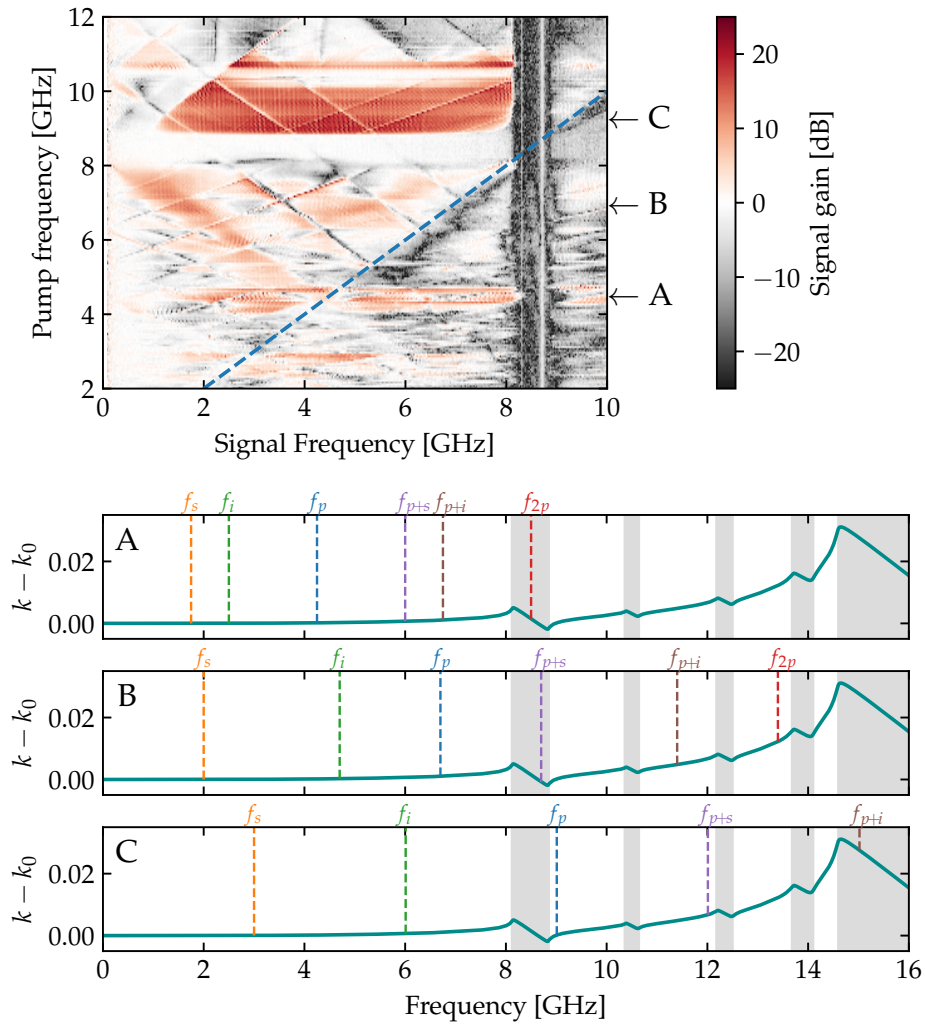


Fig. 4.16.: Three-wave-mixing parametric amplification for different pump frequencies. Four-wave mixing is negligible due to vanishing Kerr nonlinearity at this particular flux point. The dashed blue line marks $f_s = f_p$. Phase plots are calculated using the PCM model. Grey regions in the phase plots depict stopbands. The first three super-period stopbands are not shown due to their negligible effect. Depending on the position of frequencies, various 3WM parametric processes are enhanced or suppressed, as illustrated in three examples having different pump frequencies. A: The pump is at a frequency such that its second harmonic falls into the first stopband and is strongly suppressed, enabling moderate signal gain. B: For frequencies where either f_{p+s} or f_{p+i} fall into the first stopband, the signal gain is enhanced. C: For pump frequencies in a broad range above the first stopband, phase-matching is achieved between signal, idler, and pump, while higher sideband frequencies are mismatched or fall into the second stopband. This regime features high broadband signal gain.

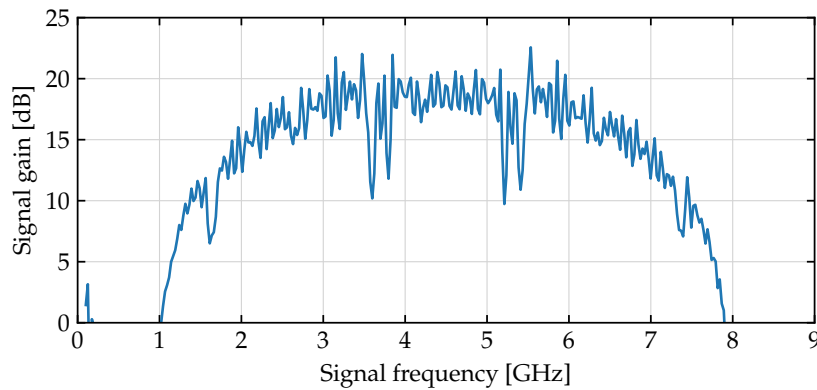


Fig. 4.17.: Broadband three-wave-mixing gain. The gain profile was measured at 4.2 K with a pump frequency of 9.01 GHz and an approximate pump input power of -57 dBm.

coherence length $\xi = \pi/\Delta k \gg N$ (Sec. 2.4.5). Under this condition, exponential signal growth is possible as the signal propagates along the TWPA. If the coherence length is short, $\xi < N$, the growth is hindered. If one of the involved frequencies coincides with a stopband, its wave is evanescent, i.e., it cannot propagate, and phase-matching to other frequencies is not possible. Such a process is most effectively suppressed. This is the case for the second-harmonic generation of the pump in Example A of Fig. 4.16, or for f_{p+s} in Example B. Both cases lead to moderate signal gain, since signal power leakage to these suppressed tones is avoided. Other processes, however, are not suppressed, so the signal gain is limited.

High and broadband signal gain by phase-matched downconversion is possible for a range of pump frequencies between ca. 9–10 GHz. This is enabled by several conditions, as illustrated by Example C of Fig. 4.16. First, the second harmonic of the pump falls into the second stopband, which ranges from ca. 14.5 to 20 GHz, so that its propagation, and hence, its growth, are strongly suppressed. Therefore, power leakage of the pump to its harmonics is avoided. Second, the higher sideband frequencies, f_{p+s} and f_{p+i} , produced by 3WM upconversion processes, are either phase-mismatched due to the strong curvature of the dispersion relation, or fall into the second stopband. In either case, their growth is suppressed, too, and signal and idler power leakage to these tones is avoided. In this way, multiperiodic capacitance variation enables efficient broadband parametric amplification. The amplification performance is analyzed in the following.

4.5.2 Phase-matched three-wave mixing downconversion

A representative gain spectrum is shown in Fig. 4.17. The average gain in the center of the spectrum is ca. 18 dB, close to the envisaged gain of 20 dB. The nearly elliptical

shape of the gain spectrum suggests phase matching of 3WM downconversion at its center, and increasing phase mismatch beyond. Ignoring ripple and the distinct gain dips, the 3-dB bandwidth of the average gain is ca. 4.5 GHz, ranging from 2.2 to 6.7 GHz, which is more than one octave of instantaneous bandwidth. However, this bandwidth is interrupted by deep gain dips and has strong ripple of more than ± 2 dB at some frequencies.

Ripple is an unwanted but expected feature in the gain profile of any TWPA [50, 69, 38]. They are a manifestation of Fabry-Perot interferences caused by interfering multireflections of the back- and forth-propagating signal waves between the input and the output of the TWPA. The same occurs for the idler, which, nonlinearly coupled to the signal, leads to signal gain ripple as well. Multireflections are usually attributed to impedance mismatch between the TWPA and its environment, having a nominal impedance of $50\ \Omega$. In a TWPA with periodic capacitance variation, however, there is another mechanism leading to multireflections even in the absence of a nominal impedance mismatch. Indeed, the TWPA analyzed here has an (average) impedance of $\bar{Z} = \sqrt{L_{\text{cell}}/\bar{C}} = 47\ \Omega$ at the applied flux bias, too close to $50\ \Omega$ to explain the strong ripple. Instead, the sidelobes of the stopbands cause a frequency-dependent impedance Z and lead to pronounced reflections [94], see Sec. 3.3.3. The latter can be seen as a quasi-frequency-periodic pattern in the $|S_{11}|$ plot of Fig. 4.8, calculated by the corresponding CTM model.

A close-up view on the region of phase-matched 3WM downconversion, presented in Fig. 4.18, gives insight into the origin of the gain ripple and the dips. Several superposed patterns can be observed. Two fine-structured patterns are superposed, one with a vertical orientation and another with a diagonal orientation parallel to the cyan dashed line. This line marks the frequency where the idler is at the edge of the stopband, indicating involvement of the idler. These patterns are Fabry-Perot interferences of the signal and idler, respectively, caused by aforementioned sidelobes. The idler is parametrically coupled to the signal, so if the idler experiences constructive or destructive interference, the signal is also affected. Fine-tuning of the pump frequency allows selecting conditions where the ripple is rather moderate. Apart from ripple, the gain profile is remarkably insensitive to pump frequency variations.

The deep dips in the gain spectrum have a different origin. In Fig. 4.18 they are visible as traces of integer slopes -1 and $+2$, marked by dashed and dash-dotted black lines, respectively. At these frequencies, the sideband frequencies f_{p+s} and f_{p+i} coincide with the fifth, sixth, or seventh super-period stopband at frequencies $f_{\text{sp},j}$ with index $j = \{5, 6, 7\}$, see Sec. 4.2.3. The super-period stopbands with $j = \{1, 2, 3\}$ are not pronounced (Fig. 4.8) and do not have any noticeable effect. In cases where f_{p+s} coincides with a stopband at $f_{\text{sp},j}$, the signal gain has a distortion

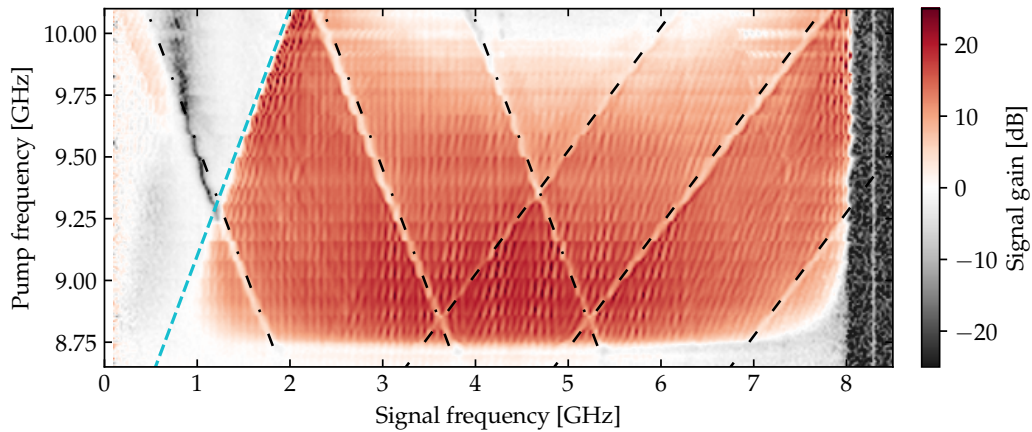


Fig. 4.18.: Region of phase-matched 3WM. The cyan dashed line marks where the idler frequency is at the edge of the stopband. The dashed (dashdotted) lines, having integer inverse slopes -1 ($+2$), indicate the signal frequencies at which f_{p+s} (f_{p+i}) coincide with one of the super-period stopbands caused by the unwanted periodicity $4m$ (see Fig. 4.3). Two superposed interference patterns can be observed: the Fabry-Perot interference patterns of the signal (vertical pattern) and the idler (tilted pattern).

at $f_s = f_{sp,j} - f_p$, having a slope of -1 between f_s and f_p . If $f_{p+i} = f_p + f_i = 2f_p - f_s$ coincides with $f_{sp,j}$, the signal gain is distorted at $f_s = f_{sp,j} + 2f_p$, giving a slope of $+2$ between f_s and f_p . In this way, the CPW 180° turns, periodically interrupting the arrays of the TWPA, lead to severe gain distortion, manifesting as deep dips and high peaks in the gain spectrum.

4.5.3 Comparison to simulations

Further insight into the behavior and dynamics of nonlinear effects occurring in the TWPA is given by circuit simulations. The circuit model of the TWPA, discussed in Sec. 4.2.1 and with circuit parameters listed in Tab. 4.4, was employed for circuit simulations in WRspice. A comparison of the experimental gain spectrum and the corresponding simulation is presented in Fig. 4.19. Since the WRspice model does not feature accurate modeling of dielectric loss by lacking frequency-dependent resistors, the pump power of the simulation was kept as a free parameter to roughly match the shape of the experimental gain curve. Simulation and experiment are in good agreement, and the general shape as well as the distinct features such as ripple and gain distortions are reproduced well, illustrating the accuracy of the circuit model.

Circuit simulations make power levels of waves propagating inside the TWPA accessible. The spatial evolution of the signal, idler, and sideband waves is shown in Fig. 4.20. The real part of the complex power is plotted for each mode (see

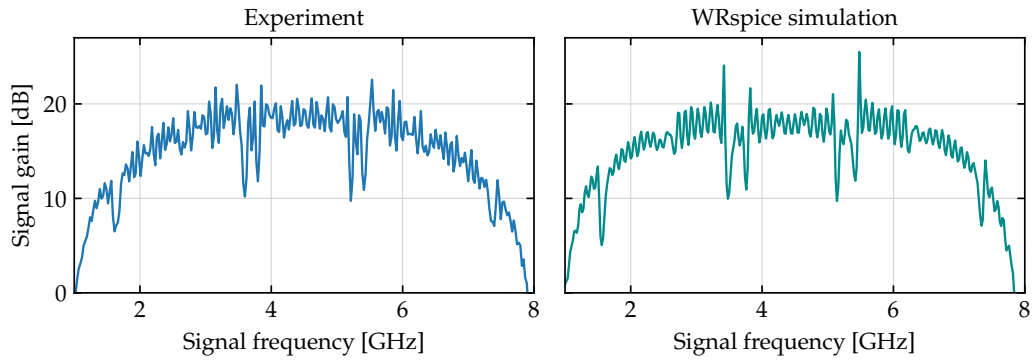


Fig. 4.19.: Three-wave-mixing gain profile. Comparison between experiment (left) and circuit simulation (right).

Sec. 3.1.3). At the signal frequency, marked by letter A in the gain spectrum, the signal and idler waves are increasing almost exponentially between node 1000 and the output, node 2352. The signal input power in the simulations is -100.5 dBm. Towards the input of the TWPA, node 0, the power of the reflected, back-propagating, signal wave is higher than that incident on the TWPA, causing an interference dip around node 800. Due to the strong back-propagating wave, $|S_{11}| > 0$ dB for some frequencies, which can be detrimental for sensitive signal sources such as qubits. For this reason, in practical qubit readout schemes with TWPAs, isolators are placed between the qubits and the TWPA. Interference of the back-propagating with the incident signal wave produces gain ripple.

The sideband tone with frequency f_{p+i} grows with strong undulations, indicating a coherence length of ca. 250 nodes due to phase mismatch, after which the direction of power transfer reverses. Still, substantial power is transferred between the idler and the sideband f_{p+i} , such that the latter is only 5 dB lower than the idler, and the idler wave deviates notably from exponential growth. This circumstance leads to the conclusion that the maximum achievable gain of the TWPA is limited, among other reasons, by power leakage to unwanted parametric processes. A possible measure to increase the achievable gain is to increase the variation depth ζ_2 of the multiperiodic capacitance variation scheme. Initially, a value of $\zeta_2 = 0.85$ was targeted, but the neglected stray capacitances (Sec. 4.1) of predominantly the smallest capacitance value C_{01} lead to a reduced value of $\zeta_2 = 0.588$.

The spatial evolution of the waves at frequencies marked by letters B and C illustrates the circumstances of the gain distortions caused by the super-period stopbands. In B, the sideband tone f_{p+s} is strongly enhanced, such that its amplitude is partially larger than those of signal and idler. The same applies for the sideband tone f_{p+i} in C. Power leakage leads to the gain dips.

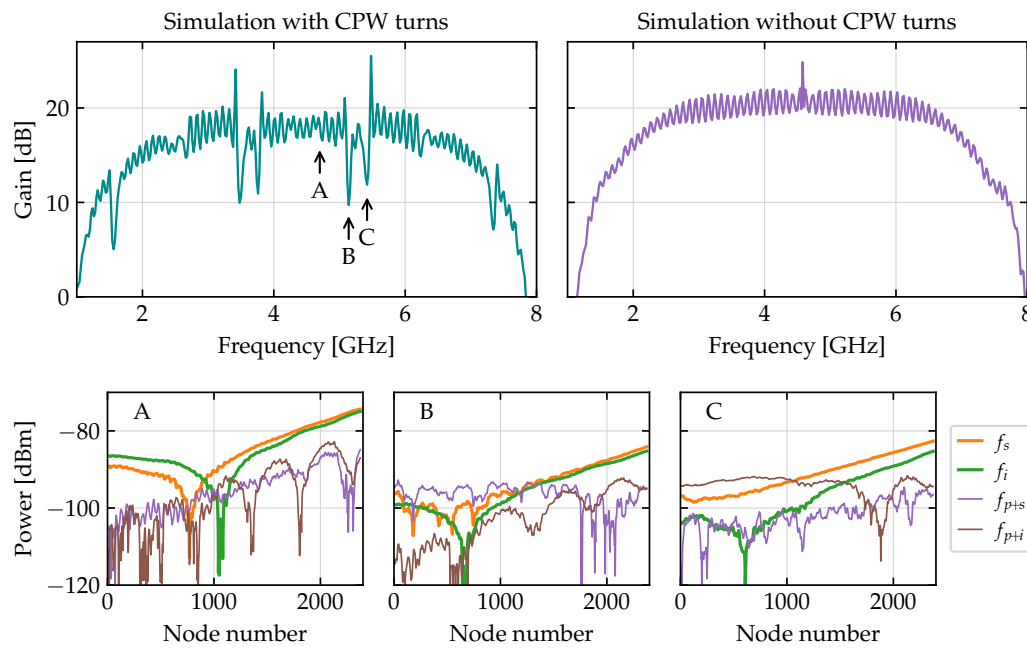


Fig. 4.20.: Simulations of the gain profile with and without CPW segments. Removing the periodic CPW segments from the TWPA leads to a gain spectrum without severe gain distortions. Ripple is still present. Simulation conditions are equal, apart from the pump frequency. The bottom panels give insight into the spatial evolution of the waves of signal, idler, and the first two sideband tones for the case with CPW segments. The input of the TWPA is at node 0, and the output at node 2352. The signal input power in the simulations is -100.5 dBm.

To investigate the TWPA performance without the unwanted super-period stopbands, the circuit simulation model was modified, removing the transmission-line segments corresponding to the periodic CPW 180° turns. The resulting gain spectrum (top-right panel in Fig. 4.20) does not suffer from gain distortions caused by super-period stopbands. Their absence slightly increased the width of the first stopband, requiring a higher pump frequency to achieve phase-matched parametric amplification. As a side effect, the average gain is ca. 2 dB higher. Pump powers and all other conditions were equal in the simulations. The moderate but still present ripple stems from Fabry-Perot interferences caused by the sidelobes of the stopbands, as discussed above. This simulation gives evidence that the undesired gain distortions can be avoided by filling the CPW 180° turns with unit cells, effectively making the array uniform without periodic interruptions by CPW segments.

4.5.4 Gain saturation

An important figure of merit of an amplifier is its saturation power, stating the upper bound of the linear range of the amplifier. Typically, the saturation power of an amplifier is defined as the signal power at which the gain decreases by 1 dB, also called the 1-dB compression point. It is determined by increasing the signal power gradually while measuring the signal gain.

In the characterization of superconducting parametric amplifiers it is a challenge to calibrate the absolute power at the sample input, setting the reference plane into an inaccessible cryogenic environment. Standard microwave measurement techniques, such as a power calibration using commercially available power sensors, is not possible in cryogenic setups. The insertion loss and gain of the input and output lines are temperature dependent and are equally unknown, so measuring the roundtrip transmission fails for power calibration. Instead, a physical process producing a reference power level inside the cryogenic environment can be used.

We use the power-dependent inductance shift of the rf-SQUIDs of the TWPA to calibrate the input power, as discussed in Sec. 4.3. This places the reference plane to the input of the TWPA, perfectly suited for the measurement of input-referred saturation power, and yields a reasonable uncertainty. Moreover, the frequency range at which the calibration is valid, ca. 4–4.5 GHz, coincides with the center of the gain spectrum, where the saturation power is measured. The measurements for the input power calibration were performed in the same cooldown as the saturation power measurement.

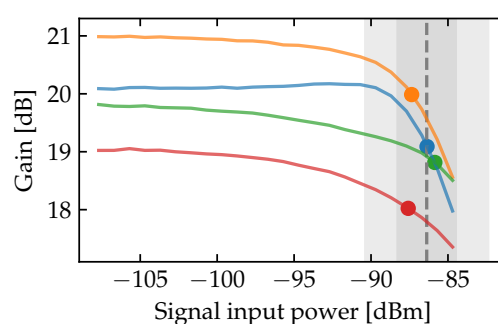


Fig. 4.21.: Gain saturation. Four representative curves with a low-power gain of 19–21 dB are shown. Different saturation mechanisms are at play, leading to different 1-dB compression points (circles), i.e. the power where the gain is reduced by 1 dB. The dashed line marks the mean value of 36 such points. The inner shaded area is one standard deviation. The outer shaded area is one standard deviation plus the uncertainty of the input power calibration. The saturation power at a gain of 20 dB is (-86.5 ± 4) dBm.

In the literature on superconducting parametric amplifiers, the saturation power is typically stated at a gain of 20 dB for comparability. Therefore, we measure the

gain spectrum for increasing signal powers, and evaluate the gain saturation at frequencies where the low-signal-power gain is in the range of 19–21 dB. Figure 4.21 presents four representative curves. Different mechanisms can cause gain saturation (Sec. 3.4.1), which explains why the trajectories of the presented curves differ, and gain compression sets in at different signal powers for different frequencies. The 1-dB compression points at 36 frequencies between 3 and 6 GHz have a mean value of -86.5 dBm and a standard deviation of 1.9 dB. Considering the uncertainty of the input power calibration, estimated to be 2.1 dB, the saturation power of the TWPA, at a gain of 19–21 dB, is (-86.5 ± 4) dBm.

This is, to the best of my knowledge, the highest saturation power obtained in a Josephson-junction-based TWPA, and is exceeded only by kinetic-inductance based TWPAs [38, 7, 160, 112], c.f. Table 4.6 in Sec. 4.7. Compared to other reported Josephson-junction-based TWPAs with similar gain [49, 50, 76, 53], this saturation power is about one order of magnitude larger. At the same time, the number of unit cells, $N = 2352$, is not significantly larger than those of Refs. [49, 50] with 2037 and 2160 unit cells each.

This is possible because in an rf-SQUID-based TWPA the parameter β_L , representing a degree of freedom in the design, allows reducing the nonlinearity without changing the linear inductance, while their ratio is fixed when using Josephson junctions as nonlinear inductive elements (see Sec. 3.4). In the same manner, the (linear) lead inductance L_f further dilutes the nonlinearity (Eq. 4.6). Due to the reduced nonlinearity, the required pump power to achieve similar gain is larger, so it can amplify higher signal powers before it saturates. Plugging in the circuit parameters of Tab. 4.4 to the model presented in Sec. 3.4.4, the predicted saturation power for a gain of 20 dB is -90.5 dBm at the Kerr-free point. On the other hand, applying the estimate of Eq. 3.50 and the experimental pump power of ca. -57 dBm yields a saturation power of -83 dBm. The almost 8 dB difference between these estimates can be attributed to the high insertion loss of ca. 7 dB at the pump frequency, not accounted for in the theoretical model. Although both theoretical predictions lie within the experimental uncertainty interval, the discrepancies are not yet fully understood.

4.5.5 Four-wave-mixing regimes

The flux-tunability of the TWPA enables not only 3WM, but also 4WM parametric amplification. In contrast to 3WM, this is possible in two different regimes, determined by the sign of the Kerr nonlinearity. At zero magnetic flux (and at full flux quanta) the Kerr nonlinearity is positive, while at fluxes of half flux quanta the Kerr nonlinearity is negative (see Fig. 4.9). The Kerr nonlinearity produces a power-

dependent phase shift due to self-phase modulation and cross-phase modulation, resulting in a power-dependent phase mismatch in 4WM parametric amplification. In the design of a 4WM TWPA it is therefore a major task to correct this phase mismatch. In the 3WM TWPA presented in this work, certain regimes exist where 4WM leads to significant broadband parametric amplification, although the device was not designed for such operation regimes and is not impedance matched at such flux bias values.

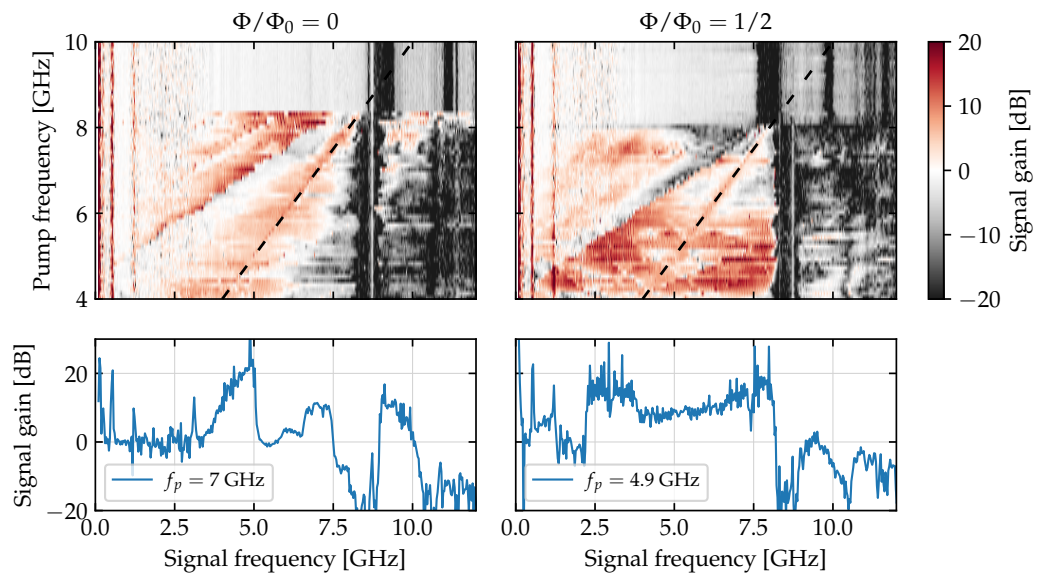


Fig. 4.22.: Four-wave mixing regimes with positive and negative Kerr nonlinearity. The fingerprint of 4WM is a gain profile centered around the pump frequency (dashed lines). 3WM gain is not observable. Left panels: Regime of positive Kerr nonlinearity at zero flux bias. Phase matching occurs when the pump frequency is placed slightly below the gap. Right panels: Reversed-Kerr regime at a flux bias of $\Phi_0/2$. The self-phase and cross-phase modulations compensate the chromatic dispersion, leading to broadband phase-matched gain with the typical camelback shape. The impedance of the TWPA is mismatched to $50\ \Omega$ in both regimes. The stopband is significantly shifted by cross-phase modulation in opposite directions.

Figure 4.22 illustrates 4WM parametric amplification in both regimes of positive and negative Kerr nonlinearity. The fingerprint of the principal 4WM process, obeying frequency relation $f_s + f_i = 2f_p$, is a gain profile centered around the pump frequency. In the case of positive Kerr nonlinearity, $\Phi = 0$, this process is not inherently phase-matched due to the nonlinear phase mismatch, and gain is limited to values on the order of 10 dB for the present device. However, when the pump frequency is placed slightly below the first stopband, the wavenumber of the pump is increased, compensating the nonlinear phase mismatch. Under this condition, phase-matching can be achieved and high gain exceeding 20 dB is possible (left gain

spectrum in Fig. 4.22). This is the operation principle of dedicated 4WM TWPAs such as those presented in Refs. [49, 50].

Considering the case of negative Kerr nonlinearity, at a flux $\Phi = \Phi_0/2$, the nonlinear phase mismatch partially compensates the linear phase mismatch given by chromatic dispersion, because the two have reversed signs. At certain pump powers, phase-matching can be achieved. This enables 4WM gain in a very broad range of signal frequencies, and features the distinct camelback-shaped gain profile of the reversed-Kerr TWPA [53].

4.6 Noise characterization

One of the key characteristics of a TWPA is its remarkable noise performance approaching the limits set by quantum mechanics in a several GHz wide range of frequencies. Hence, tenfold improvement of the signal-to-noise ratio (SNR) of a measurement system is possible when the first amplifier in the chain is a TWPA [50]. How to measure the added noise of such a system in a wide range of frequencies, however, remains an area of ongoing research.

For experiments such as reading out qubits, the noise of the whole system is the relevant noise characteristic, as it determines the achievable SNR. The minimum system noise of an amplifier apparatus is one photon [5], called the standard quantum limit (SQL). To further improve a measurement setup, it is an important aspect of noise characterization to identify and quantify the different sources of noise within the system responsible for system noise above the SQL. These contributions encompass external sources introduced by the environment surrounding the TWPA, as well as the added noise of the TWPA itself.

This section describes the wideband noise characterization of a system where a TWPA is the first amplifier stage. The obtained system noise is as low as 2.5–3.5 photons in a bandwidth of a few GHz, on par with other state-of-the-art TWPA setups. Furthermore, the different contributions to the system noise are identified and strategies to mitigate them are discussed.

4.6.1 Overview on noise measurement techniques

Noise measurement is an absolute measurement, in contrast to relative measurements such as S-parameter measurements. Narrowband noise measurements are possible by calibrating the absolute input power in the cryogenic system, e.g., via the ac Stark shift of a qubit [49, 77]. Such methods usually allow noise measurement only at a single frequency. Most wideband noise measurements use a wideband calibrated noise source, supplying a known amount of noise to the system under test. Commonly used wideband noise sources are matched loads (resistors) placed at different fixed temperature levels of the cryostat (hot/cold loads) [129, 161], heated loads with variable temperatures [38, 162, 53, 124], and shot-noise tunnel-junctions (SNTJs) [112, 118, 124]. Variable-temperature loads can be fabricated using commercially available components [162] and are easy to use. Their noise temperature depends on the physical temperature of the load. The noise temperature of an SNTJ depends on a voltage bias applied, which is tunable much faster than the temperature of variable-temperature loads, where thermal equilibrium has to be awaited. On the other hand, the SNTJ requires a bias tee, inserting additional loss

to the system, and strong magnetic fields to suppress superconductivity of the junction, which may be incompatible with other components in the cryostat.

The key challenge in wideband noise characterization of systems containing a TWPA arises from multi-mode interactions in the wide bandwidth of the TWPA. The noise source applies a known noise level to the input of the system, while the output noise is measured, typically with a spectrum analyzer. However, a wideband noise source not only applies noise at the signal frequency (at which the system output noise is measured), but also illuminates other frequencies, such as the corresponding idler frequency, and sideband frequencies. All these modes interact parametrically and determine the system output noise. Hence, the noise model has to account for this increased effective input noise. Malnou et al. have pointed out that the noise figures reported in several previous works, Refs. [38, 163, 129], did not account for the idler input noise, although they employed wideband noise sources such as hot and cold loads. This results in an underestimation of the added noise by about a factor of two [112]. In the following, I follow roughly the approach of noise modeling and measurement as described by Malnou et al. [112, 112, 124], but extend their two-mode description to the multi-mode case.

Very recently, Klimovich et al. [161] have used a wideband noise source to measure only the noise of the elements after the TWPA, and therefore have circumvented the need of considering the idler (and possibly, higher sidebands) in the noise measurement. Although this technique requires a measurement setup with a switch to selectively connect a noise source or the TWPA to the measurement setup, the advantages of this approach could dominate over the extra source of uncertainty introduced by the different switch paths.

4.6.2 Noise measurement setup with a variable-temperature load

Variable temperature platform

In this work, a heated $50\ \Omega$ load, terminating a coaxial line to the TWPA input, is used as a calibrated noise source. This load is thermally anchored on a variable-temperature platform (Fig. 4.23), designed and fabricated in-house, and consisting of a copper plate thermally insulated against the cold plate of the cryostat and having a thermal weak link to the mixing-chamber plate of the cryostat. The platform temperature is set by a heater and monitored by a calibrated Ruthenium oxide temperature sensor, the latter of which is positioned in close proximity to the $50\ \Omega$ load. The thermal weak link allows a compromise between settling times of platform temperatures and unwanted heating of the mixing-chamber plate. At a platform temperature of 1.8 K (4 K) the mixing-chamber plate temperature increases

to ca. 60 mK (300 mK). For platform temperatures up to 1 K the effect on TWPA performance by changing its temperature is assumed to be negligible.

Noise of the variable-temperature load

The emitted noise of a resistor at temperature T is the Johnson-Nyquist noise. A generalized form of the Johnson-Nyquist noise, also valid for low temperatures $k_B T \ll hf$, was formulated by Callen and Welton [165], and can be understood as a variant of the dissipation-fluctuation theorem [166]. Here, k_B is the Boltzmann constant, and h is the Planck constant. If the resistor is impedance-matched to the remaining circuit, it emits a noise power

$$P_{\text{JN}}(f, T) = k_B T B \frac{hf / (k_B T)}{\exp\left(\frac{hf}{2k_B T}\right) - 1} + \frac{hfB}{2} \quad (4.38)$$

$$= \frac{hfB}{2} \coth\left(\frac{hf}{2k_B T}\right) \quad (4.39)$$

at a frequency f and in a bandwidth B . The classical limit of this equation is $P_{\text{JN}} = k_B T B$, valid for $k_B T \gg hf$, and yields a constant power spectral density, i.e., white (frequency-independent) noise. For low temperatures $k_B T \ll hf$, the limit is $P_{\text{JN}}/B = hf/2$, reflecting the quantum-mechanical zero-point fluctuations of half a photon. In the literature, noise power is also often expressed as a noise temperature $T_N = P/k_B B$. Particularly for low noise powers it is convenient to express noise in terms of numbers of photons, $N = P/hfB$, so Eq. 4.39 reads

$$N_{\text{JN}}(f, T) = \frac{1}{2} \coth\left(\frac{hf}{2k_B T}\right). \quad (4.40)$$

Noise measurement setup

The full noise measurement setup is shown in Fig. 4.23. The variable-temperature platform is mounted to the cold plate (ca. 100 mK) of the cryostat, but due to the Teflon insulation to the cold plate and the thermal weak-link to the mixing chamber plate, the base temperature of the platform is ca. 44 mK. The variable-temperature load is connected to the rest of the setup via a superconducting coax cable to keep the loss between noise source and TWPA as small as possible. Unavoidable insertion loss in the path before the TWPA is placed by a bias tee and a directional coupler, which are components necessary to bias and pump the TWPA.

The pump is supplied to the TWPA via the coupling port of the directional coupler, having 20 dB coupling coefficient, which attenuates the thermal noise of higher-temperature stages coming through the pump line. To bring the thermal

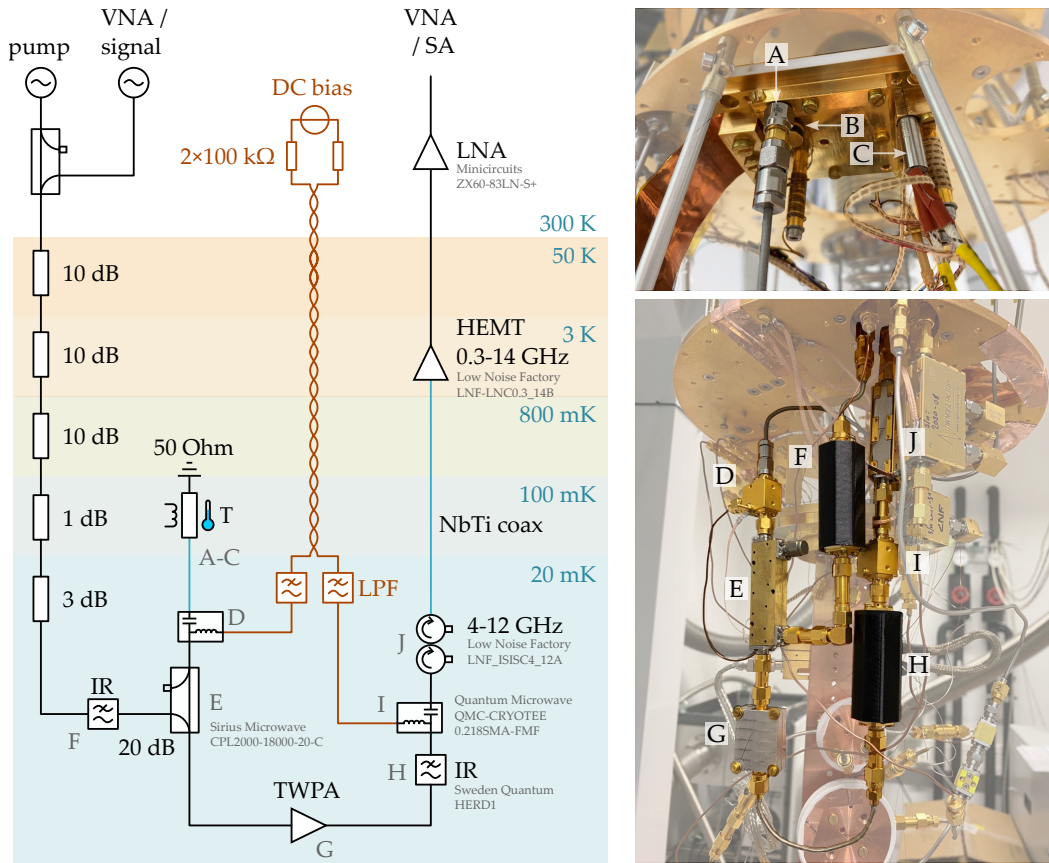


Fig. 4.23.: Measurement setup for noise characterization. The noise source is a heated 50 Ω termination (A), thermally anchored at a variable-temperature platform with a temperature sensor (B) and a heater (C). To minimize insertion loss, the only components between the TWPA sample and the noise source are the bias tee (D) and the directional coupler (E), and the coax cables between the noise source and the bias tee (D), as well as between the isolator (J) and the HEMT, are superconducting. Low-pass filters (LPF) on the DC lines are 1.5 m long twisted Thermocoax® lines between the 300 K flange and the mixing chamber plate (Sec. 4.4). The IR filters on the coaxial lines are leaky-waveguide filters (F, H) [164]. The housings of all components are thermally anchored on the mixing-chamber plate by copper strands. The voltage source for DC bias is a National Instruments NI-9269.

noise from room temperature components down to levels close to the vacuum noise, heavy attenuation is placed at various temperature stages of the cryostat. Calculating the thermal-photon populations of the pump line at the different stages (using Eq. 4.42), the thermal noise from higher temperature stages incident on the TWPA for frequencies above 3 GHz is at least one order of magnitude smaller than the half-photon vacuum fluctuations of the noise source at base temperature. This justifies the assumption that the noise at the input of the TWPA is determined only by the noise from the noise source, plus the Johnson-Nyquist noise induced by the insertion loss of the bias tee and the directional coupler.

After the TWPA, a dual-junction isolator prevents noise and reflections from the HEMT to propagate back to the TWPA. Moreover, infrared filters protect the TWPA from infrared radiation leaking through the coax lines and the dc lines to avoid excess quasiparticles. A cryogenic HEMT amplifier and a room-temperature low-noise amplifier (LNA) further amplify the signal (and noise) to power levels easily processable with room-temperature electronics like a spectrum analyzer.

4.6.3 Noise model

Amplifiers and attenuators

Every amplifier not only amplifies the input power, but unavoidably also adds some noise. When this added noise N_{added} is referred to the amplifier's input (as a matter of choice), the total output noise is

$$N_{\text{out}} = G(N_{\text{in}} + N_{\text{added}}), \quad (4.41)$$

where N_{in} denotes the input noise incident to the amplifier input, see Fig. 4.24a. Quantum mechanics demands that the added noise of any linear phase-insensitive amplifier is at minimum half a photon [5]. Together with the half photon of zero-point fluctuations, incident on the input of the amplifier, the minimum amount of input-referred noise of any phase-insensitive amplifier is one photon. This is called the standard quantum limit (SQL), or the one-photon quantum limit. Note that phase-sensitive (degenerate) amplification does not have this limit, but an uncertainty principle instead [5].

An attenuator, or any lossy element, does not only dissipate part of the input power, it also adds Johnson-Nyquist noise, depending on the temperature of the lossy element (Fig. 4.24b). Noise in an attenuator is analogous to a beamsplitter interaction, where the input signal and the Johnson-Nyquist noise represent the

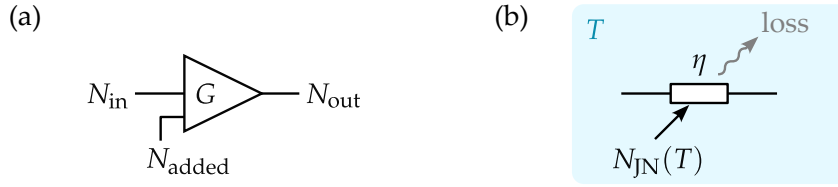


Fig. 4.24.: Noise models of an amplifier and an attenuator. **(a)** An amplifier amplifies the input power by gain G and adds input-referred noise N_{added} . **(b)** An attenuator at temperature T attenuates by transmission efficiency η and adds input-referred Johnson-Nyquist noise $(1 - \eta)N_{\text{JN}}$.

incident beams, and the output power and dissipated power correspond to the outgoing beams. Hence, the output noise of an attenuator at temperature T is

$$N_{\text{out}} = \eta N_{\text{in}} + (1 - \eta)N_{\text{JN}}(T), \quad (4.42)$$

with the power transmission efficiency $\eta = 10^{-A/10}$, where A is the attenuation in dB.

Single-mode, two-mode, and multi-mode amplifiers

Equation 4.41 is valid for amplifiers which are not parametric, such as, e.g., the HEMT. These amplifiers amplify each input tone independently, and are thus in the following called single-mode amplifiers. Parametric amplifiers, on the other hand, rely on parametric interactions between multiple tones.

In an ideal parametric amplifier, only the three tones signal, idler, and pump, are involved. When the pump tone is assumed constant in amplitude and free of noise (due to its high amplitude), an ideal parametric amplifier is effectively a two-mode amplifier (Fig. 4.25b). It is symbolized in the following by two input (output) ports, one for the signal tone and the other for the idler tone, although the device has only one physical input (output) port. Input power at the signal frequency also leads to output power at the idler frequency, and vice versa, since the two frequencies are coupled parametrically. When the signal is amplified by a power gain G , the photon-conversion gain from the idler frequency at the input to the signal frequency at the output is $G - 1$ (Eq. 2.99), normalized by the ratio of the energies of the involved photons. Thus, additional to the amplified signal photons $G N_{\text{in},s}$, $N_{\text{in},i}$ idler photons at the input convert to $(G - 1)N_{\text{in},i}$ signal photons at the output,

$$N_{\text{out},s} = G N_{\text{in},s} + (G - 1)N_{\text{in},i} \quad (4.43)$$

$$\approx G(N_{\text{in},s} + N_{\text{in},i}), \quad (4.44)$$

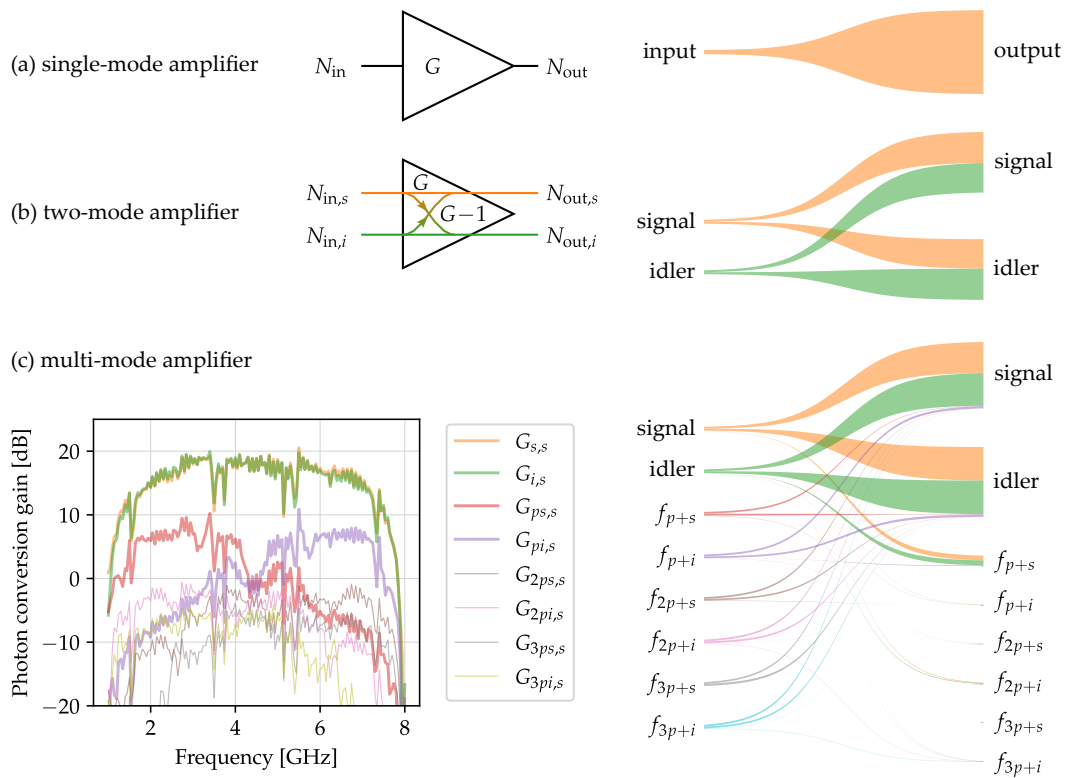


Fig. 4.25.: Illustration of noise composition in single-mode, two-mode, and multimode amplifiers. **(a)** In a single-mode amplifier, all tones are amplified independently. **(b)** An ideal parametric amplifier is a two-mode amplifier, where the signal and idler tones are parametrically coupled. The two inputs symbolize the signal and idler modes. Input noises at both frequencies contribute to the signal output noise $N_{out,s}$. **(c)** Multimode dynamics in a realistic TWPA. Left plot: Photon-energy-normalized conversion gains for various input tones and the signal output, simulated using WRspice with the circuit model of the TWPA (conditions like in Fig. 4.19). Right: Noise composition illustrated as a flow diagram, showing the energy flow of eight tones for $f_s = 4$ GHz and $f_p = 8.9$ GHz, and equal input noise levels of $N_{in,j} = 1/2$ photon each. The signal output noise contains contributions not only of the signal and idler input noise, but also from all sidebands.

where Eq. 4.44 is the limit for high gain, $G \gg 1$. This two-mode behavior is illustrated in Fig. 4.25b as a Sankey-like power-flow diagram for equal input noises $N_{\text{in},s} = N_{\text{in},i} = 1/2$.

An important consequence of Eq. 4.44, when compared to Eq. 4.41, is that the amplified idler input noise appears as added noise of the amplifier. The minimum input noise at each frequency are vacuum fluctuations, $N_{\text{in},s} = N_{\text{in},i} = 1/2$. Dividing the resulting output noise, given by Eq. 4.44, by the signal gain G , the input-referred output noise is one photon, the standard quantum limit. Therefore, this fundamental limit manifests in the case of an ideal parametric amplifier as the input vacuum noise at the signal and the idler frequencies.

Due to its broadband nature, a realistic TWPA is a multi-mode amplifier and enables parametric processes of many involved frequencies, far beyond the two-mode model. Figure 4.25c illustrates the multimode dynamics of our TWPA for the first eight modes, being the signal, the idler, and the first six sidebands. In the left panel, the photon-conversion gains $G_{m,s}$, where m is the index of the mode with frequency f_m , are plotted against the signal frequency. The conversion gains are extracted from WRspice simulations, using the same circuit model and parameters as in Fig. 4.19. For each signal frequency, eight simulations were carried out, probing the circuit with each of the eight input modes independently. The power-conversion gain for input mode f_n to output mode f_m is

$$G_{m,n}^p = \frac{|b_{2,m}|^2}{|a_{1,n}|^2}, \quad (4.45)$$

with the power-wave quantities as defined in Eqs. 3.4, 3.5. Normalizing for photon energies hf yields the photon-conversion gain

$$G_{m,n} = \frac{|b_{2,m}|^2}{|a_{1,n}|^2} \frac{f_n}{f_m}, \quad (4.46)$$

stating how many photons of input mode f_n are converted to photons of output mode f_m . Power-conversion gains correspond to X-parameters, a nonlinear extension of S-parameters. Photon-conversion gains correspond to the respective quantum X-parameters [104, 105], introduced by Peng et al. [99] for the multi-mode description of TWPAs.

Considering multi-mode interactions only, neglecting contributions due to losses, the resulting signal output noise is

$$N_{\text{out},s} = \sum_m G_{s,m} N_{\text{in},m} = G \left(N_{\text{in},s} + \frac{G_{s,i}}{G} N_{\text{in},i} + \sum_{m \in \{ps, pi, 2ps, \dots\}} \frac{G_{s,m}}{G} N_{\text{in},m} \right). \quad (4.47)$$

Note that the signal photon gain $G_{s,s}$ is identical to the signal power gain G . How the output noise is composed in our TWPA at a frequency of $f_s = 4$ GHz is illustrated in the right panel of Fig. 4.25c with an input noise of $N_{in,m} = 1/2$ across the first eight modes m . Input noise of all modes contribute to the signal output noise. At the same time, a considerable amount of information contained in the signal is lost to the unmeasured sidebands, in the example particularly to f_{p+s} .

The multi-mode model, Eq. 4.47, has two main differences to the ideal two-mode model, Eq. 4.43. First, the idler conversion gain $G_{s,i}$ may differ from $G - 1$. Second, input noises at the sideband frequencies contribute to the signal output noise and are thus a source of excess noise above the standard quantum limit.

In practice, it is difficult to experimentally determine conversion gains in a cryogenic environment, because it requires calibration of absolute power at a reference plane at the input or output of the amplifier in a wide range of frequencies. Moreover, in the case of the present TWPA, the influence of the sidebands on the output noise is rather small due to strong sideband suppression by multiperiodic capacitance variation. Therefore, in the following we rely on a modified two-mode model in the evaluation of the experimental data. Nevertheless, I want to emphasize the general importance of sidebands in wideband noise characterization of TWPAs. Neglecting sidebands and deviations of the idler conversion gain from G introduces systematic errors in the commonly used Y-factor method, see Sec. 4.6.4.

Two-mode model of a nonideal parametric amplifier

A realistic parametric amplifier may have an idler conversion gain deviating from $G - 1$, and adds excess noise N_{exc} above the standard quantum limit, having various possible origins. Neglecting sideband-induced noise for now leads to the simplified two-mode model

$$N_{out,s} = G \left(N_{in,s} + \frac{G_{i,s}}{G} N_{in,i} + N_{exc} \right). \quad (4.48)$$

This equation is used for evaluating noise measurement data in the next section. The excess noise incorporates sideband-induced noise, Johnson-Nyquist noise due to dissipation within the TWPA (mostly by dielectric loss), phase noise of the pump, flux noise of the flux bias (or, alternatively, current noise of the current bias, etc.). A detailed investigation of all these effects is beyond the scope of this thesis and remains a future prospect to further decrease the achievable system noise.

System noise model: pump off

A simplified schematic of the noise measurement apparatus is given in Fig. 4.26a (complete schematic see Fig. 4.23). If the pump is off, the TWPA can be seen as

a passive component attenuating by transmission efficiency η_T . All components between the TWPA and the noise source can be combined by transmission efficiency η_1 . All components subsequent to the TWPA can be modeled by an effective (single-mode) amplifier with gain G_2 and added noise N_2 . An effective schematic is shown in Fig. 4.26b. Consecutively applying Eqs. 4.41 and 4.42, the output noise of the system is

$$N_{\text{out}} = G_2 \eta_T \eta_1 \left(N_{\text{in}} + \frac{1 - \eta_1}{\eta_1} N_f + \frac{1 - \eta_T}{\eta_1 \eta_T} N_f + \frac{N_2}{\eta_1 \eta_T} \right) \quad (4.49)$$

where $N_f = 1/2$ are the vacuum fluctuations in the lossy components at 20 mK and in the relevant frequency range. Equation 4.49 can be simplified to that of an effective amplifier,

$$N_{\text{out}} = G_{\text{sys}}^{\text{off}} (N_{\text{in}} + N_{\text{sys}}^{\text{off}}), \quad (4.50)$$

later used for fitting experimental data. Then, the gain and noise of the elements after the TWPA are

$$G_2 = \frac{G_{\text{sys}}^{\text{off}}}{\eta_T \eta_1} \quad (4.51)$$

$$N_2 = \eta_1 \eta_T \left(N_{\text{sys}}^{\text{off}} - \frac{1 - \eta_1}{\eta_1} N_f - \frac{1 - \eta_T}{\eta_1 \eta_T} N_f \right). \quad (4.52)$$

Shifting the reference plane to the output of the TWPA (dashed purple line in Fig. 4.26b), the system output noise is

$$N_{\text{out}} = G_2 (N_{\text{out}}^T + N_2). \quad (4.53)$$

where N_{out}^T is the output noise of the TWPA.

System noise model: pump on

When the pump is on, the TWPA is modeled as a two-mode amplifier using Eq. 4.48, see Fig. 4.26c. The reference plane is at the output of the noise source, such that all components relevant for TWPA operation (bias tee and directional coupler) are included to the system under test. The system noise determined in this way is the relevant quantity for an experiment where a signal source, e.g., a qubit, replaces the noise source.

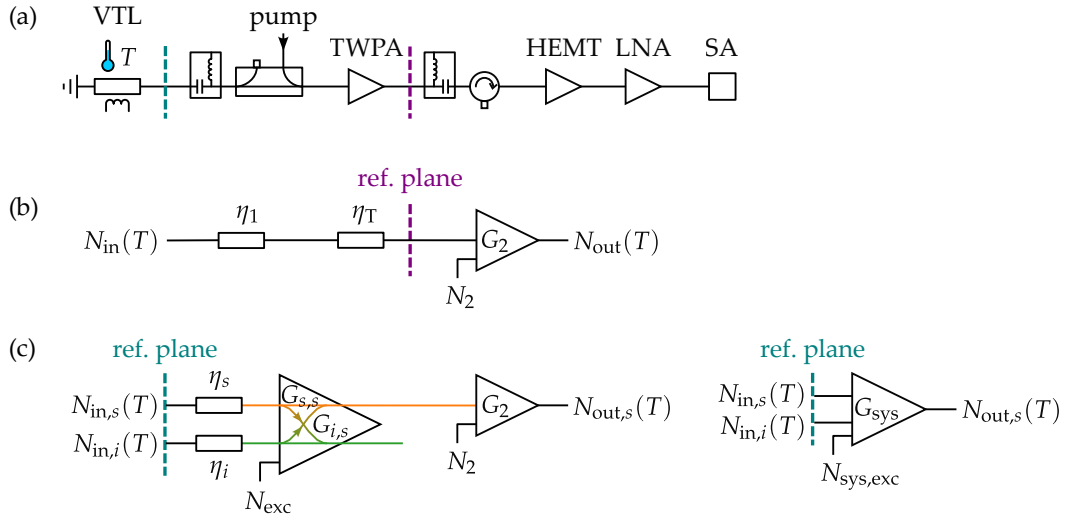


Fig. 4.26.: Noise modeling with a variable-temperature load (VTL) as noise source. **(a)** Simplified schematic of the noise measurement system. The VTL supplies calibrated noise $N_{in}(T)$ to the system. The output noise $N_{out}(T)$ is measured with a spectrum analyzer (SA) at the signal frequency. **(b)** Schematic when the pump is off. The elements before the shifted reference plane (purple dashed line), including the unpumped TWPA, have transmission efficiency η_1 . **(c)** Schematic when the pump is on, with reference plane at the VTL, supplying input noise $N_{in,s}$ ($N_{in,i}$) at the signal (idler) frequency to the system, attenuated by η_s (η_i) by the elements before the TWPA. The TWPA is modeled as a two-mode amplifier with excess noise N_{exc} . The overall system has gain G_{sys} and excess noise $N_{sys,exc}$.

The lossy elements before the TWPA attenuate the input noise from the noise source by $\eta_{1,s}$ at the signal frequency and by $\eta_{1,i}$ at the idler frequency. Using Eq. 4.42, the noise incident on the TWPA is

$$N_{in,s}^T = \eta_{1,s} \left(N_{in,s} + \frac{1 - \eta_{1,s}}{\eta_{1,s}} N_f \right), \quad (4.54)$$

$$N_{in,i}^T = \eta_{1,i} \left(N_{in,i} + \frac{1 - \eta_{1,i}}{\eta_{1,i}} N_f \right). \quad (4.55)$$

Using Eq. 4.48, the output noise of the TWPA at the signal frequency is

$$N_{out,s}^T = G\eta_{1,s} \left(N_{in,s} + \frac{1 - \eta_{1,s}}{\eta_{1,s}} N_f + \frac{N_{exc}}{\eta_{1,s}} \right) + G_{i,s}\eta_{1,i} \left(N_{in,i} + \frac{1 - \eta_{1,i}}{\eta_{1,i}} N_f \right) \quad (4.56)$$

where all TWPA excess noise is attributed to the signal-to-signal path. Applying the approximation $\eta_{1,i} \approx \eta_{1,s} \approx \eta_1$, and including the elements after the TWPA using Eq. 4.53, we find the total system noise

$$N_{out,s} = G_2 G \eta_1 \left(N_{in,s} + \frac{G_{i,s}}{G} N_{in,i} + 2 \frac{1 - \eta_1}{\eta_1} N_f + \frac{N_{exc}}{\eta_1} + \frac{N_2}{G \eta_1} \right). \quad (4.57)$$

The last three terms represent the noise contributions exceeding the standard quantum limit under the conditions of gain symmetry $G_{i,s}/G = 1$ and input noises being vacuum fluctuations. These contributions to the system noise are further analyzed in Sec. 4.6.6. Combining them as the system excess noise $N_{\text{sys,exc}}$ and identifying system gain $G_{\text{sys}} = G_2 G \eta_1$, Eq. 4.57 simplifies to

$$N_{\text{out},s} = G_{\text{sys}} \left(N_{\text{in},s} + \frac{G_{i,s}}{G} N_{\text{in},i} + N_{\text{sys,exc}} \right). \quad (4.58)$$

which is the equation used for fitting experimental data, see next section.

4.6.4 System-noise measurement using the Y-factor method

A widely used method to measure the added noise of an amplifier is the "Y-factor" method. The output noise power is measured when the amplifier is connected to a 'hot' noise source and when it is connected to a 'cold' noise source. The ratio of these two noise powers, known as the Y-factor, allows quantifying the amplifier's added noise and gain using Eq. 4.41.

Y-factor method for cryogenic and parametric amplifiers

To apply this method to a cryogenic amplifier, some modifications are necessary. First, the output of the amplifier under test, e.g. the parametric amplifier, is not directly accessible, so only the whole amplification chain can be characterized. Second, the noise of a noise source, with $k_B T$ not necessarily much larger than hf , does not linearly increase with its temperature and has to take into account quantum corrections, see Eqs. 4.38–4.40. Third, if the amplifier under test is parametric, the input noise at the idler frequency has to be accounted for.

The Y-factor method in such a case is illustrated in Fig. 4.27a–c. For a range of temperatures of the load, i.e., known effective input noise levels (Panel a), the output noise (Panel b) is measured and fitted to a noise model (Panel c). The noise $N_{\text{in}}(f, T)$ from the load heated to temperature T is given by Eq. 4.40. Three cases of the dependence of the effective input noise on the temperature of the load are illustrated in Panel d: the effective input noise of a single-mode amplifier (Eq. 4.41), of an ideal two-mode amplifier (Eq. 4.44), and of a multi-mode amplifier (Eq. 4.47), each evaluated for $f_s = 4$ GHz and $f_p = 8.9$ GHz. The conversion gains of the idler and the sidebands increase the effective input noise.

The theoretical output noise of a system with a parametric amplifier as first stage is depicted in Panel b, using Eq. 4.58 assuming gain asymmetry $G_{i,s}/G = 1$, and with parameters given in the figure caption. The curve is normalized to the system gain and, therefore, the y-intercept of the curve corresponds to the system noise.

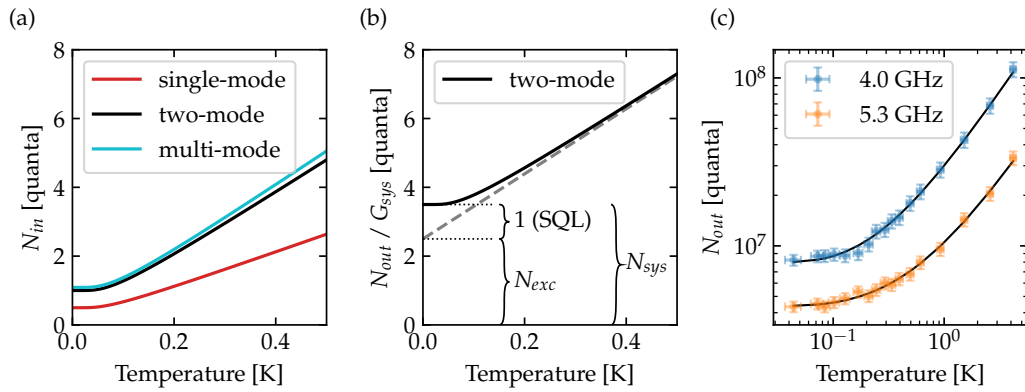


Fig. 4.27.: Noise measurement using a variable-temperature load (VTL) as noise source. **(a)** Dependence of the effective input noise of the amplifier under test on the temperature of the VTL for a single-mode (pump off), two-mode (pump on), c.f. Fig. 4.26b,c, and a multi-mode amplifier, with $f_s = 4$ GHz and $f_p = 8.9$ GHz. At 0 K the input noise is the vacuum fluctuation of 1/2 photon (1 photon) for a single-mode (two-mode) amplifier. **(b)** Theoretical input-referred output noise $N_{sys}(T) / G_{sys}$ of a system with a two-mode amplifier as first stage. In the example, $G_{sys} = 10^7$ and excess noise $N_{sys,exc} = 2.5$ photons above the standard quantum limit (SQL). The y-intercept of the curve yields the system noise. Dashed line: Classical (high-temperature) limit. **(c)** Experimental data for two frequencies and fits with a two-mode model.

Panel c shows experimental data and fits to Eq. 4.58 for two exemplary frequencies. In general, a noise measurement is possible using only two temperature levels, e.g., using a hot and a cold load [38, 161], but the accuracy greatly benefits from measuring the output noise in a range of temperatures and fit it to a noise model.

Caveats and systematic errors in the Y-factor method

The Y-factor method has several caveats when applied to cryogenic and parametric amplifiers with very small added noise [167, 112, 53]. If the classical limit of the Johnson-Nyquist noise, $N_{in}(f, T) = k_B T / hf$ (dashed line in Fig. 4.27b), is used instead of the full expression of Eq. 4.40, the obtained system noise is underestimated by the amount of the standard quantum limit, one photon, in the case of a two-mode amplifier. Furthermore, disregarding the idler's contribution to the effective input noise in the case of a parametric amplifier leads to a systematic error of about a factor of two. The resulting system noise is underestimated by that factor, while the system gain is overestimated. Since a factor of two in power gain corresponds to 3 dB, this overestimation can be easily overlooked, given that typical system gains are often on the order of 60–80 dB.

Consequently, neglecting the contributions of the sidebands to the effective input noise introduces a systematic error in the same way (Fig. 4.27a). Their contribution

to the added noise of our TWPA is in the range of 5–25% of the standard quantum limit, as simulations suggest for frequencies in the range 1–7.5 GHz (Fig. 4.25c). Peng et al. report 19% [99] for the TWPA of Ref. [49]. A possibly larger effect is the gain asymmetry $G_{i,s}/G = 1$, which can be in the range of ± 3 dB in a TWPA, e.g., due to loss asymmetries at the signal and idler frequencies [168, 124].

Another systematic error is possible by gain saturation of the parametric amplifier (or any other amplifier in the system) due to high input noise levels [53]. The systematic error occurs because the resulting system gain is not constant but depends on the input noise level. Gain saturation can be taken into account by a modified noise evaluation scheme [124]. A yet to be investigated systematic error source is the power dependence of dielectric loss at millikelvin temperatures and few-photon power levels [169, 84], at which noise measurements on parametric amplifiers typically take place. Similar to gain saturation, this power dependence of dielectric loss also results in a power dependence of the system gain.

4.6.5 Results

Data acquisition and fit

For a range of temperatures of the load, the output power of the system was measured with a spectrum analyzer both with the pump on and off. From the obtained power spectrum $P(f)$, photon numbers are calculated as $N_{\text{out},s} = P/(hfB)$, where $B = 1$ MHz is the resolution bandwidth of the spectrum analyzer. To reduce the randomness of the measured noise power spectral density, 20 traces were averaged each measurement, and the video bandwidth was set to 10 kHz, far below the resolution bandwidth, which has the effect of increasing the sampling time and therefore stabilizing the result.

The data is then fitted to Eq. 4.50 (pump off) and Eq. 4.58 (pump on). For the latter, data and fitted curves for two exemplary frequencies are shown in Fig. 4.27c. To prevent systematic errors by TWPA gain saturation, in particular in the peaks of the gain ripples, only temperatures < 1 K of the load were taken into account for the fit of the "pump on" data. Error bars of the data points comprise a power uncertainty of 0.3 dBm, reflecting residual randomness of the data acquisition, and a propagated temperature error, accounting for temperature drift during data acquisition and thermometry uncertainty. For each temperature step, thermal equilibrium was awaited before the measurement was triggered. During the measurement, the temperature was stable within 5%. The temperature sensor is an individually calibrated Ruthenium oxide sensor with a temperature-dependent uncertainty. Both temperature uncertainties are propagated to an uncertainty $\Delta N_{\text{in}}(T) = |\partial N_{\text{in}}/\partial T|\Delta T$ to be accounted for in the fitting algorithm. The impedance of the variable-temperature

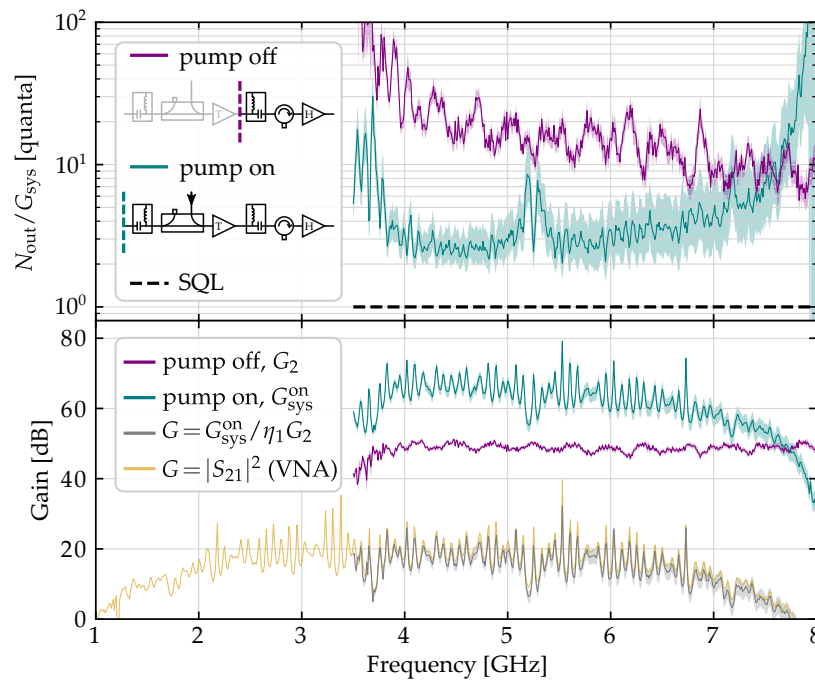


Fig. 4.28.: Characterization of system noise and gain. Top panel: Input-referred system noise, when the HEMT (pump off) or the TWPA (pump on) is the first amplifier in the chain, in comparison to the standard quantum limit (SQL). Pump off: Only the elements after the TWPA are considered via shifting the reference plane (purple dashed line), the system gain is G_2 . Pump on: The system noise is a few quanta above the SQL, the system gain is $G_{\text{sys}}^{\text{on}} = G_2 G \eta_1$. Bottom panel: System gains for pump off and on, and the resulting TWPA gain in comparison to the VNA measurement. Shaded areas show uncertainties dominated by the fit (pump off) and the gain asymmetry (pump on), see main text.

load is assumed to be sufficiently well matched to the coaxial line to the sample, such that the corresponding mismatch loss is negligible.

System gain and noise

The results of the fits are shown in Fig. 4.28 as the input-referred system noise and the system gain. For the spectra where the pump is off the reference plane was shifted to after the TWPA using Eqs. 4.51 and 4.52. The out-of-band insertion loss of the isolator, placed between the TWPA and the HEMT, limits the frequency range of the noise measurement to ca. 3.5–12 GHz. Therefore, not the full bandwidth of the TWPA can be characterized.

The system noise of the elements after the TWPA, N_2 , characterized by the "pump off" measurement, corresponds to a noise temperature of ca. 3–5 K for frequencies above 4 GHz, only slightly above the specified value of the HEMT from its datasheet. This shows that the HEMT noise is dominating the noise of the

elements after the TWPA, which is expected. Other important contributions to N_2 are the lossy elements between TWPA and HEMT, and the noise of the spectrum analyzer. The shaded area represents the fitting uncertainty discussed above.

When the TWPA is pumped, the system noise decreases by almost an order of magnitude to levels down to 2.5 quanta, i.e., 1.5 quanta above the standard quantum limit. This result is comparable to system-noise levels of other state-of-the-art TWPAs characterized with a broadband noise source [162, 53, 124]. A detailed comparison to literature values is given in Sec. 4.7. Note that the system noise also depends on the lossy components before the TWPA and, in comparison to typical setups for qubit readout, our setup does not contain a (lossy) isolator before the TWPA. This can explain why our system noise is slightly lower than those reported in some other works.

The shaded area of the "pump on" measurement represents uncertainties arising from the fitting procedure, the unknown gain asymmetry, and the unknown contribution of sidebands. The latter two, identified as sources of systematic error and discussed in Sec. 4.6.4, are accounted for by considering gain asymmetry $G_{i,s}/G = \pm 3$ dB. This value may overestimate the true gain asymmetry, as simulations suggest a value not larger than 1–2 dB for most frequencies (Fig. 4.25c). In this way, it also accounts for the contribution of sidebands, which have a similar effect as the gain asymmetry. The sum of the first six sideband conversion gains (last term in Eq. 4.47) ranges from 0.03 to 0.27 photons in the signal band 1–7.5 GHz. Summing up the input noise of the idler and the sidebands yields 0.4–0.8 photons, which compares to half a photon of signal input noise, or $-1... + 2$ dB of effective gain asymmetry. In order not to rely on the validity of these simulations, we consider a larger value of ± 3 dB for the error bars. Note that gain asymmetry and sideband conversion gains are difficult to experimentally determine due to the need for wideband input-power calibration at a cryogenic reference plane.

The distinct double-peak in the spectrum of the system noise corresponds to the gain dips caused by sideband frequencies close to super-period stopbands, discussed in Sec. 4.5.2. Their effect on noise is further discussed in the next section. Moreover, ripples in the noise correspond to ripples in the system gain, stemming mostly from Fabry-Perot interferences by multireflections. The gain ripple in this measurement is substantially larger than that measured in the dipstick at 4.2 K (Fig. 4.17). We attribute this to a loose contact in an SMA connector subsequent to the TWPA, which was detected only after the cooldown. Additional reflections then cause enhanced Fabry-Perot interferences within the TWPA. This loose contact also explains the ripple in the curves where the pump is off.

The TWPA gain inferred from subtracting system gain with pump on and off (considering the shifted reference plane), is on average ca. 1.8 dB lower than that

measured by the VNA (grey and golden lines in Fig. 4.28). This discrepancy can be explained by an offset in the VNA calibration, which was performed in a different cooldown. Other possible reasons are underestimated insertion loss before the TWPA, and deviations from the model used for the Y-factor fit. Note that the former has no influence on the system noise.

4.6.6 Contributions to system noise

The relevant noise benchmark of a measurement apparatus for, e.g., reading out qubits, is the system noise, being the noise performance of the whole system and determining the resulting SNR. The system noise includes contributions from the losses of the elements preceding the TWPA, the added noise of the TWPA itself, and contributions from the elements subsequent to the TWPA. Identifying and quantifying these contributions indicates paths for further optimization of the amplification chain.

Neglecting the gain asymmetry in Eq. 4.57 for simplicity, and assuming vacuum noise at the TWPA input, $N_{\text{in},s} = N_f$, by well-thermalized components before the TWPA, the system noise is

$$N_{\text{sys}} = 2N_f + 2\frac{1-\eta_1}{\eta_1}N_f + \frac{N_{\text{exc}}}{\eta_1} + \frac{N_2}{G\eta_1}. \quad (4.59)$$

The first term, $2N_f = 1$ photon, represents the standard quantum limit, the second term corresponds to the noise due to the lossy elements before the TWPA, N_{exc} is the excess noise of the TWPA beyond the quantum limit, and the last term represents the noise contribution of the elements subsequent to the TWPA. The attenuation of the directional coupler and the bias tee is ca. 0.9 dB, based on device specifications, and yields $\eta_1 = 0.81$. For the TWPA gain G the VNA measurement is taken, as it presents an upper bound for the TWPA excess noise, compared to the gain obtained from the system noise measurement, as described above. Finally, N_{exc} is obtained by rearranging Eq. 4.59. All four contributions to the system noise are depicted in Fig. 4.29 as a stacked plot.

TWPA excess noise

For our amplification chain, the TWPA excess noise clearly dominates the noise exceeding the quantum limit. This noise has been attributed to two potential sources: dielectric loss [49] and the generation of sidebands [98].

The latter is described by Eq. 4.47, where for neglected signal-idler gain asymmetry the term $\sum_{m \in \{ps, pi, 2ps, \dots\}} N_f G_{s,m} / G$ represents the sideband-induced excess noise, assuming vacuum fluctuations N_f at the input for all sideband frequencies

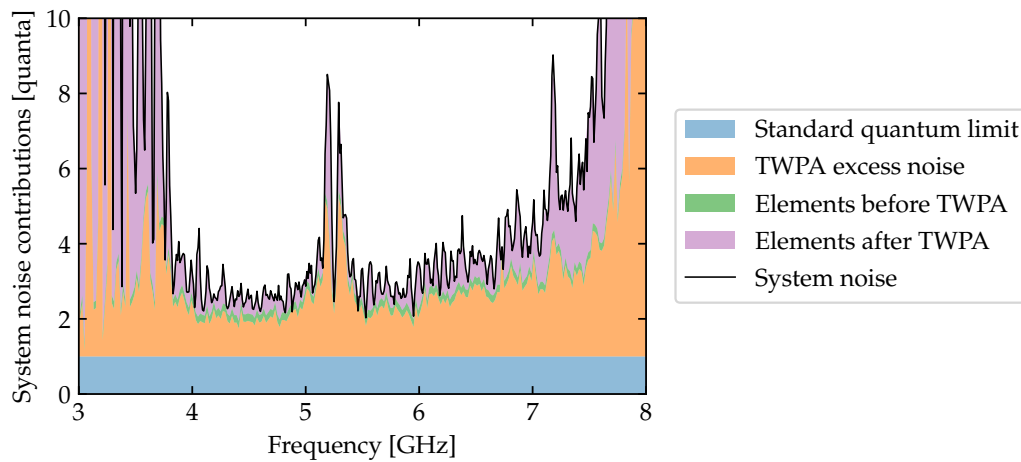


Fig. 4.29.: Stacked plot of contributions to the system noise. TWPA excess noise is the deviation from an ideal parametric amplifier with noise equal to the standard quantum limit. Noise contributions from the elements after the TWPA are small where the TWPA gain is large. It can be overcome by higher TWPA gain. Noise from the lossy elements before the TWPA cannot be overcome by higher TWPA gain.

f_m . Note that in Fig. 4.25 only the input noise was accounted for, but there is also some possibly non-negligible coupling between backward-traveling waves of noise incident on the output of the TWPA and the forward-propagating signal wave [98]. Considering all 14 non-ideal contributions (vacuum noise at 6 sidebands incident on the input, and at signal, idler, and the sideband frequencies incident on the output of the TWPA) in the simulation, the sideband-induced noise ranges from 0.03 to 0.28 photons in the signal band. For most signal frequencies, sideband-induced noise is therefore only a small contribution to the TWPA excess noise.

Nevertheless, at the signal frequencies where the first sidebands are in close vicinity of the super-period stopbands causing the distinct gain dips, the sideband-induced noise is strongly increased. This happens because at these frequencies, sideband generation is enhanced, and therefore, input noise at the sidebands contributes more to the signal output noise. We attribute the increased excess noise in the whole vicinity of the two distinct peaks to this effect, which is most notable for frequencies between 5 and 5.5 GHz in Fig. 4.29, as well as at 3.5–4 GHz and around 7.2 GHz. Consequently, eliminating super-period stopbands has great potential to further decrease the TWPA excess noise.

The second source, dielectric loss, likely has a greater impact on the excess noise of the TWPA. It originates from Johnson-Nyquist noise due to dielectric loss, in the same way as discussed for the case of an attenuator in Sec. 4.6.3. Therefore, the dielectric loss distributed over the TWPA array does not only attenuate propagating

signals, it also partially replaces signal with noise, and can be seen as an internal noise source inside the TWPA. Since Johnson-Nyquist noise (vacuum fluctuations when $k_B T \ll hf$) is broadband, it further leads to signal output noise via multimode interactions.

A rigorous quantum-mechanical treatment of the (asymmetric) loss in TWPAs and its effect on noise was done by Houde et al. [168], considering a two-mode model with signal and idler. If sideband conversion gain is significant, the two-tone distributed gain-loss model may underestimate the loss-induced noise. This is because the broadband Johnson-Nyquist noise not only occurs at signal and idler frequencies, but also at the sidebands, which then cause increased signal noise via parametric coupling. Including multimode interactions, beyond a two-mode model, requires a much more involved numerical approach, which was realized by Peng et al. [98, 99] applying the harmonic balance method. I want to emphasize that the same multimode approach is also possible with WRspice time-domain simulations, but is much more computation-costly than the harmonic balance method, which leads to estimated simulation times of years rather than minutes for a full treatment of an 8-tone analysis.

Contributions of other system elements

The system noise also includes contributions from the elements before and after the TWPA. The latter, $N_2 / (G\eta_1)$, indicates that the TWPA is almost sufficient to overcome the noise of all subsequent amplifier stages. However, there is still room for slight improvement in the system noise by achieving a few dB higher TWPA gain. The former, on the other hand, cannot be overcome by higher gain. Typically, the elements before the TWPA are isolators, preventing back-action of the TWPA to the (sensitive) signal source, and devices like directional couplers and bias tees, needed to supply the pump wave and bias current to the TWPA. Reducing their contributions to the system noise is possible by either choosing less lossy commercially available components, or by replacing them by superconducting on-chip passive microwave components. Another upcoming field of research is the development of nonreciprocal TWPAs, which amplify in forward direction and isolate in backward direction, potentially making the isolator between signal source and TWPA redundant.

4.7 Summary and comparison to the state of the art

In summary, in this chapter I have presented design, modeling, circuit parameter extraction, and amplification and noise performance of an rf-SQUID based TWPA with multiperiodic capacitance variation. Possible improvements have been identified with the help of circuit simulations. These include proper plate capacitor design accounting for stray capacitance to hit the targeted stopband frequency, increasing the capacitance variation depth ζ_2 to achieve higher gain, and filling the CPW 180° turns with unit cells to avoid super-period stopbands and their detrimental effect on gain flatness. Future prospects are replacing the SiO_x dielectric by a low-loss dielectric, such as SiN_x or amorphous silicon, to reduce the TWPA excess noise, and apply apodization techniques to decrease sidelobes and hence gain ripple.

For comparison to the state of the art in TWPA development, Table 4.6 compiles reported TWPA devices, their architecture, and their figures of merit. Note that the definitions of these figures of merit are not standardized and the values given here may vary from those stated in the references. In the table, gain and 3-dB bandwidth are estimated from reported gain curves ignoring ripple, the latter of which are stated separately. In most works, the saturation power is stated as an input-referred 1-dB compression point for an unsaturated gain of ca. 20 dB. At lower gain the saturation power is higher, so the gain at which the saturation power is measured is also stated. Also, it is important to note that the methods for noise measurement vary significantly, which limits the comparability of the quoted system noise values.

The TWPA reported in this thesis is comparable to other state-of-the-art TW-PAs in gain, ripple, bandwidth, and system noise. Outstanding is its saturation power, which is about one order of magnitude higher than that of other reported Josephson-junction based TWPAs when compared at a gain of 20 dB. Various reported saturation powers are plotted in Fig. 4.30 against their pump power. Kinetic-inductance-based TWPAs offer much higher saturation powers, which comes at the price of orders of magnitude higher pump powers. This usually requires pump cancellation techniques to avoid saturating subsequent amplifier stages with the pump. Regarding noise performance, the very recently reported TWPA by Klimovich et al. [161] is worth mentioning, which employs amorphous silicon as a low-loss dielectric and achieved the lowest system noise in the presented overview. Finally, the TWPA by Ranadive et al. [53] has achieved the lowest ripple at (on/off) gains of approximately 20 dB. This design achieves phase-matching without stopband engineering.

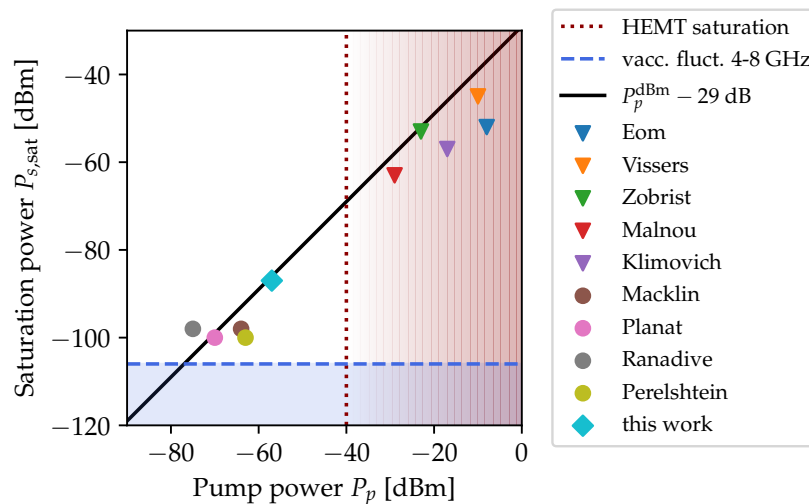


Fig. 4.30.: Saturation powers of reported TWPAs. Circles denote Josephson-based TWPAs and triangles denote kinetic-inductance-based TWPAs. The TWPA of this work (diamond) exhibits ca. 10 dB higher saturation power than other Josephson-junction-based TWPAs. Pump cancellation techniques are necessary if the pump power of the TWPA exceeds the saturation power of the subsequent HEMT amplifier (vertical dotted line). The red-shaded area marks the region where heating of the lowest-temperature stage of the cryostat by the pump is likely (μW range). A lower limit on required TWPA saturation power is given by the power of vacuum fluctuations in the TWPA bandwidth, shown for a gain bandwidth of 4 GHz. For references see Table 4.6.

Tab. 4.6.: Summary of reported TWPAs and their figures of merit for kinetic inductance based TWPAs (top) and Josephson-junction-based TWPAs (bottom).

Year, first author	mixing process	nonlinear element	material class	phase-matching	gain, ripple [dB]	bandwidth [GHz]	saturation power [dBm] (at gain)	pump power [dBm]	system noise [quanta] (at frequency)
2012 Eom [38]	4WM	KI CPW	NbTiN	PL	10 ± 5	4	-52 (18 dB)	-8	3.4 ^a (9.4 GHz)
2014 Bockstiegel [7]	4WM	KI CPW	NbTiN	PL	27 ± 5 ^b	6		-10	
2016 Visser [160]	3WM	KI CPW	NbTiN	PL	20 ± 5	4	-45	-10	
2016 Adamyan [45]	4WM	KI QF	NbN	PL	6 ± 2			-15	
2017 Chaudhuri [110]	4WM	KI FB	NbTiN	PL, RPM	16 ± 3	3		-10	
2018 Ranzani [163]	3WM	KI CPW	NbTiN	PL	10 ± 2	7		-30	3.5–17 ^a (6–10 GHz)
2019 Zobrist [129]	3WM	KI FB	NbTiN	PL	15 ^c	5 ^c	-53 (15 dB)	-23	2.1 ^a (5.7 GHz)
2020 Goldstein [46]	4WM	KI MS	WSi	None	10	1.5		-53	
2021 Malnou [112]	3WM	KI FB	NbTiN	PL	16.5 ± 1.3	2	-63 (17 dB)	-29	3.1 (3.5–5.5 GHz)
2023 Kern [170]	4WM	KI QF	NbTiN	PL	10 ± 4	6			
2023 Klimovich [161]	3WM	KI FB	NbTiN	PL	20 ± 5	3	-57 (20 dB)	-17	1.4 ^d (4–7 GHz)
2024 Favezani [171]	3WM	KI FB	NbTiN	PL	9 ± 4	3	-47 (9 dB)	-39	
2015 Macklin [49]	4WM	JJ	Nb	RPM	25 ± 2	2	-99 (22 dB) ^f	-64	2.1 (5.98 GHz)
2015 White [111]	4WM	JJ	Al	RPM	12 ± 5 ^b	4	-92 (12 dB) ^b		2.4–8 ^e (3–8 GHz)
2017 Zorin [75]	3WM	rf-SQUID	Nb	None	11 ^g	3 ^c			
2019 Miano [158]	3WM	sym. rf-SQUID	Nb	None	12.5 ^g	4			
2020 Planat [50]	4WM	dc-SQUID	Al	PJJV	18 ± 3	2.3	-100 (20 dB)	-70	1.7 ^h (6 GHz)
2021 Simbierowicz [162]	4WM	JJ	Nb	None	11 ± 2	1.4		-69	2.5–6 (5–7 GHz)
2022 Ranadive [53]	4WM	SNAIL	Al	RK	19 ± 1 ^{g,i}	1.3 ⁱ	-98 (20 dB)	-75	4.7 ⁱ (6–7 GHz)
2022 Perelshtein [76]	3WM	SNAIL	Nb	PL	23 ± 2 ^{g,i}	1.2 ⁱ			3–6 ⁱ (9–10.5 GHz)
2023 Qiu [90]	4WM	JJ	Nb	RPM	16 ± 2	3	-100 (20 dB)	-63	
2023 Roudsari [77]	3WM	SNAIL	Al	None, PL	20 ± 2	3.5	-98 (20 dB)	-87	2.1 (6 GHz)
2023 Kern [170]	4WM	JJ	Nb	None, PJJV	20 ± 4	1		-57	
2024 this work	3WM	rf-SQUID	Nb	mPCV	18 ± 3	4	-86.5 ± 4.0 (20 dB)	-57	2.5–5 (4–7 GHz)

Abbreviations: kinetic inductance (KI), quasi-fractals (QF), microstrip (MS), fishbone-capacitor CPW (FB), Josephson junctions (JJ), Superconducting Nonlinear Asymmetric Inductive element (SNAIL), periodic loading (PL), resonant phase-matching (RPM), periodic JJ-variation (PJJV), reversed-Kerr (RK), multiperiodic capacitance variation (mPCV).

^a Idler noise not accounted for.

^b Two cascaded TWPAs.

^c Data quoted but not shown.

^d System noise not quoted in the preprint. Estimated from TWPA added noise of ca. 0.6 photons and quoted HEMT noise.

^e At 8 dB gain, no calibrated noise source.

^f Ref. [123] states a saturation power of (-96.7 ± 2.3) dBm for this device.

^g Gain measurement as pump on/off ratio, TWPA insertion loss is not accounted for.

^h Ref. [123] states a system noise of 3 quanta for this device.

ⁱ Two gain lobes.

Conclusions and outlook

In this thesis, I investigated a three-wave-mixing traveling-wave parametric amplifier (3WM TWPA), starting from theoretical and simulation-based studies and culminating in the experimental demonstration of an rf-SQUID-based 3WM TWPA with state-of-the-art characteristics. The device features 18 dB of gain in a 3-dB bandwidth of more than 4 GHz (ignoring gain ripples), spanning from 2.2 to 6.7 GHz, a system noise of 2.5 photons (1.5 photons above the standard quantum limit), and a saturation power of (-86.5 ± 4.0) dBm. The latter is about 10 dB larger than saturation powers of other Josephson-junction-based TWPAs reported in the literature [49, 50, 53, 76, 90]. With such characteristics, this amplifier is suitable for simultaneous frequency-multiplexed readout of detector arrays or qubits. In particular, compared to other state-of-the-art TWPAs, the high saturation power allows for either an increase in the number of qubits per readout line or a decrease in intermodulation between frequency-multiplexed qubits through an increased overhead to the saturation power [131], both of which are crucial for applications with frequency-multiplexed microwave signals, e.g. superconducting-circuits-based quantum computing.

5.1 Summary and conclusions

Theory and Concept

The presentation started with a discussion and theoretical description of nonlinear inductive elements based on Josephson junctions. Following the TWPA concept proposed by Zorin [57], the element further discussed and experimentally implemented in this thesis is the rf-SQUID. A key feature of rf-SQUIDs is the possibility of in-situ tuning of both linear inductance and nonlinearity via an applied magnetic field or an injected constant current. Tuning the linear inductance allows impedance matching of the TWPA to its embedding circuitry, while the simultaneously tuned nonlinearity determines the regime of either 3WM or 4WM. Remarkably, a Kerr-free regime of pure 3WM is possible, featuring increased saturation power and large separation of the strong pump tone from the signal band. Beyond the common description of ideal rf-SQUIDs, this work extends rf-SQUID modeling to include

nonideal parasitic inductances, present in any practical implementation. The resulting linear and nonlinear properties have been derived and their impact on saturation power was discussed (Sections 2.1.3 and 3.4.4).

After presenting the linear characteristics of a one-dimensional homogeneous metamaterial based on rf-SQUIDs – including characteristic impedance, dispersion relation, and attenuation in a low-loss limit (Sec. 2.2) – the nonlinear properties of copropagating waves were described (Sec. 2.4). An overview of various parametric processes was given. When considering only three modes – the pump, signal, and idler – it is possible to achieve analytical solutions to describe parametric amplification. A key property is exponential growth of the signal amplitude with device length, when phase matching is achieved. It was shown, however, that unwanted parametric processes generate a plethora of tones in the spectrum of a 3WM TWPA with low dispersion, far beyond a three-mode model. These processes lead to power leakage from the three main tones, and prevent the TWPA from achieving exponential signal gain [58]. On the other hand, Renberg Nilsson et al. [86] have shown that in a high-dispersion regime with high gain per unit cell, these tones are suppressed, but phase mismatch prevents achieving exponential gain. Hence, it seems that a dispersion regime in a homogeneous 3WM TWPA where exponential gain is possible does not exist.

Dispersion engineering addresses both issues, and its study and implementation constitute the main focus of this thesis. Appropriately designed, it fulfills two purposes: suppressing unwanted parametric processes and maintaining phase matching of the intended parametric amplification process. Three examples of such dispersion-engineering techniques have been demonstrated in detail through circuit simulations, all relying on the sub-wavelength engineering of wave-propagation characteristics (Sec. 3.2). The first two concepts make use of plasma-frequency engineering, combined with either resonant phase matching (RPM) or periodic ground capacitance variation (PCV) to achieve phase matching. The third concept is a modification of PCV, called multiperiodic capacitance variation (mPCV), where the capacitances are varied in two periods, one to suppress unwanted parametric processes, and the other to phase-match 3WM parametric amplification. The PCV concepts do not require additional circuit elements, such as resonators in the case of RPM, which makes their experimental realization easier. It has been demonstrated that these three techniques suffice to suppress unwanted mixing processes to the extent that the multi-mode dynamics in the TWPA reduce to an effective three-mode case (c.f. Figures 3.5 and 3.7). Under such circumstances, parametric amplification can be quantitatively described by the analytical closed-form expressions derived for the ideal three-mode model.

Simulations

Another main focus of this thesis is the extensive use of time-domain circuit simulations to analyze the properties of dispersion-engineered TWPAs in the presence of multimode dynamics. For this purpose, the simulation software WRspice was employed, which contains Josephson junction models. The general approach is based on the work of Dixon et al. [58], yet significantly extends beyond it in terms of the extraction of various quantities and the complexity of the circuits under investigation. The extracted quantities include but are not limited to powers and power gains, impedances, inductances, wavenumbers, scattering parameters, power-wave quantities, and conversion gains (Sec. 3.1.3). This simulation method has been chosen because it not only avoids most of the typical approximations used in analytical or semi-analytical TWPA models, but also offers high flexibility in modeling complex circuits with nonideal elements and accounting for real-world effects such as parameter spread. Three simulation studies were presented, analyzing in detail the effects of wave reflections (Sec. 3.3), gain saturation (Sec. 3.4), and parameter spread in TWPAs (Sec. 3.5).

Signal and pump reflections, unavoidable in any practical implementation, cause gain ripples and potentially lead to instability of the amplifier (Sec. 3.3.1). A consequence with more drastic impact is the back-action of the amplifier onto the signal source, e.g. qubits, detectors, or sensors. In particular, reflected signal and idler waves can have larger amplitudes than the incident signal, potentially interfering with the signal source. Reflections can originate in impedance mismatches between the TWPA and its embedding circuitry, or in inhomogeneities within the TWPA. The use of SQUIDs (e.g. dc-SQUIDs, rf-SQUIDs, or SNAILs) allows in-situ impedance tuning via a magnetic field or bias current. However, in this work, it was demonstrated that in TWPAs utilizing periodic circuit-parameter variation, such as PCV or mPCV, the achievable impedance matching is limited by the sidelobes of the engineered stopbands (Sec. 3.3.3). A mitigation strategy has been proposed, and simulations have demonstrated its viability to suppress sidelobe-induced reflections, nearly eliminate gain ripples, and reduce backaction by more than 20 dB to values close to or below -15 dB (Sec. 3.3.4).

The signal power at which the gain starts to saturate is one of the main figures of merit of an amplifier. In 3WM TWPAs the dominant saturation mechanism is pump depletion, i.e. the reduction of the pump amplitude due to power transfer to signal and idler, whereas in 4WM TWPAs both pump depletion and signal-power-dependent phase matching due to self-phase and cross-phase modulation simultaneously cause gain saturation. Zorin [57] derived an analytical description of gain saturation in a 3WM TWPA using Jacobi elliptic functions, and this work

proposes an approximation (Eq. 3.46) resulting in a closed-form expression similar to the widely used formula by Kylemark et al. [137] for 4WM TWPAs. It highlights that 3WM TWPAs can achieve twice larger saturation powers than their 4WM counterparts for the same pump power. Based on this newly developed approximation, a workflow to design an rf-SQUID-based 3WM TWPA for a targeted saturation power was presented (Sec. 3.4.4). It agrees to within 4 dB to experimental results, with discrepancies attributed to the presence of loss in the experimental device.

The sensitivity analysis of two different dispersion-engineered rf-SQUID-based TWPA concepts on circuit-parameter spread leads to three practical implications. First, if RPM is used for dispersion engineering, the resonance frequency spread of ideally identical resonators can have a fatal impact on the TWPA performance. Strengthening the coupling between the resonators and the TWPA's transmission line somewhat mitigates this effect (Sec. 3.5.1). Second, TWPAs using PCV for phase matching do not have such a critical circuit element; therefore PCV can be considered the more robust dispersion-engineering technique regarding parameter spread. This finding has led towards the experimental implementation of a TWPA based on mPCV rather than RPM. Third, rf-SQUID-based TWPAs, biased around their Kerr-free operation point, are rather insensitive to critical-current spread of the Josephson junctions (Sec. 3.5.3), in stark contrast to TWPAs based on Josephson junctions or SNAILs as nonlinear inductive elements. Experimental evidence supporting this remarkable property is provided by the observation of convincing amplifier performance despite a large critical-current spread on the order of 20%.

Experiment

Based on the above described simulation studies, an rf-SQUID-based 3WM TWPA using mPCV for dispersion engineering was experimentally implemented and characterized. This device was fabricated in Niobium trilayer technology in a coplanar-waveguide architecture and parallel-plate capacitors with SiO_x as dielectric. A detailed circuit model was developed for deeper analysis and further optimization of future TWPA generations (Sec. 4.2). To obtain the circuit parameters for this model, we performed a number of precharacterization microwave measurements on both the TWPA itself and dedicated test chips cofabricated with the TWPA on the same wafer. Crucially, these precharacterization measurements require specialized calibration techniques suitable for cryogenic setups to accurately measure the scattering parameters of the device under test. To realistically capture the device performance, the circuit model contains various parasitic circuit elements, such as the parasitic inductances of the nonideal rf-SQUIDs, bond wire inductances,

and transmission line segments for interconnections. Linear as well as nonlinear simulation results of this model are in excellent agreement with experiments.

An issue often observed in 3WM TWPAs is insertion loss of an extent that cannot be explained by dielectric loss or quasiparticle loss. In this work, parametric frequency conversion has been identified as the responsible loss mechanism, potentially dominating over other loss mechanisms (Sec. 4.4). It was shown that frequency conversion can be driven by a noise tone of sufficient amplitude. This process is very efficient, even if the noise tone has a frequency much smaller than that of the probe tone of a transmission measurement. Transmission loss occurs due to the power transfer from the probe tone to multiple sidebands. To describe this loss mechanism, an analytical model along with a simulation model were developed. Simulations are in quantitative agreement with experimental data. The driving noise tone can either be picked up electromagnetically (e.g., via ground loops) or injected galvanically (e.g., from noisy electronic devices like a bias-current source). Therefore, special care must be taken when biasing TWPAs. Measures to prevent frequency-conversion loss include the use of low-noise current sources, twisted-pair DC wiring including low-pass filters, and high-pass filters or directional couplers on AC lines. It is noteworthy that this nonlinear loss mechanism appears as linear loss at sufficiently small probe-tone powers, so it can lead to false conclusions in the characterization of TWPAs, such as the overestimation of dielectric loss.

To determine the amplifier performance of the TWPA, the signal power gain, saturation power, and the system noise were characterized (Sec. 4.5). The range of pump frequencies for which parametric amplification is approximately phase-matched is more than one GHz wide, allowing for easy device tune-up. The average gain of 18 dB in the center of the gain profile is close to the envisaged gain of 20 dB to overcome the noise contributions of subsequent amplifier stages. The instantaneous 3-dB bandwidth, extending from 2.2 to 6.7 GHz (ignoring ripples and gain dips), represents one of the largest relative bandwidths reported in the TWPA literature, spanning more than one octave. The magnitudes of gain ripples, expected due to finite reflections, are on par with other state-of-the-art TWPAs, but the gain profile is interrupted by distortions, manifesting as gain dips. The latter arise from multireflections of higher-order mixing products at periodic coplanar-waveguide segments, and are quantitatively reproduced by circuit simulations. These coplanar waveguide segments will be avoided in the next TWPA design, eliminating the resulting gain distortions, as simulations confirm. Furthermore, circuit simulations suggest that higher signal gain may be achievable by increasing the dispersion of the TWPA further, and by using a higher pump frequency. The latter had to be

reduced to 9 GHz, compared to the targeted 12 GHz, because the stopband used for phase matching appeared at a lower frequency than originally designed.

To experimentally determine the saturation power, a method was developed for in-situ calibration of the absolute input power at the TWPA input port (Sec. 4.3). This calibration procedure utilizes the phase shift of a transmitted signal due to the power-dependent inductance of the rf-SQUIDs in the TWPA. With a power calibration uncertainty of 2.1 dB, and a standard deviation of 1.9 dB for the 1-dB compression points at different signal frequencies showing a gain of 19–21 dB, the saturation power of the TWPA is (-86.5 ± 4.0) dBm. This is about 10 dB larger than the values reported for other Josephson-junction-based TPAs at a gain of ca. 20 dB [49, 50, 53, 76, 90]. The theoretical prediction deviates by 4 dB from the experimental value, requiring additional modeling to account for the insertion loss. Yet, the large saturation power can be attributed to the low nonlinearity of the circuit, facilitated by the design of the rf-SQUIDs. These offer an additional degree of freedom, allowing to design the linear and nonlinear properties independently, in contrast to using Josephson junctions as nonlinear inductive elements. Due to the low nonlinearity, the pump power required to achieve the same gain is higher, hence it can amplify higher signal powers before it depletes.

Finally, the system noise, which is the key quantity determining the signal-to-noise ratio of the entire setup, was characterized using a variable-temperature resistive load as a wideband noise source (Sec. 4.6). A two-mode noise model was developed, based on the work of Malnou et al. [112, 172] for a modified Y-factor method suitable for cryogenic parametric amplifiers. Several caveats and systematic errors of this method were discussed, including the effects of power dependence of gain or dielectric loss. Importantly, the asymmetry of the idler-to-signal conversion gain compared to the signal gain was identified as a source for systematic error, as well as the presence of multimode parametric processes beyond the two-mode noise model. Since their experimental determination is difficult, requiring wideband absolute-power calibration at the ports of the TWPA, these two systematic errors dominate the uncertainty budgeting.

A system noise of 2.5–3 photons for signal frequencies in the ranges of 4–5 and 5.5–6.5 GHz was found, which is ca. 1.5–2 photons above the standard quantum limit, on par with other state-of-the-art TPAs characterized with a wideband noise source. Around the frequencies of the above discussed gain dips, the noise is larger. It is expected that this issue will be eliminated in the next design iteration, so that near-quantum-limited noise is achieved in a continuous broad signal band without these interruptions. Investigating the contributions to the system noise leads to the conclusion that the system noise can be further decreased by achieving a few dB higher signal gain. However, the added noise above the quantum limit

of the TWPA itself dominates. This excess noise has two major contributions. The first is sideband-induced noise, which is a rather small contribution of less than 10% for most signal frequencies, as circuit simulations suggest. Although further theoretical modeling such as a distributed noise model would be helpful, we currently conclude that the dominating source of excess noise is Johnson-Nyquist noise in the lossy dielectric.

5.2 Outlook

Two major tasks in parametric-amplifier research and development are anticipated: first, to further reduce the system noise of amplification chains with a TWPA as the first amplifier, aiming to approach the quantum limit; and second, to minimize the backaction of the TWPA on the signal source.

Rather than developing technologically sophisticated TPWAs, some developments strive to leverage resonator-based parametric amplifiers towards higher bandwidths and saturation powers to levels otherwise only achieved by TPWAs. For example, impedance-engineered resonant Josephson parametric amplifiers demonstrated bandwidths reaching 700 MHz [173, 174] and saturation powers of up to -95 dBm [51]. While practical constraints in qubit readout and other applications may prevent making full use of the several GHz wide bandwidths of TPWAs, the bandwidths of these impedance-engineered resonant amplifiers may be sufficient. Versatile narrow-band parametric amplifiers are the Dimer Josephson Junction Array Amplifiers (DJJAAs) [66], and their 3WM counterparts, the Josephson Array-Mode Amplifiers [67]. Although these devices have more narrow (tens of MHz wide) instantaneous bandwidths, the gain lobes can be flux-tuned over several GHz of dynamical bandwidth. Their lower noise compared to typical TPWAs, as well as their saturation powers comparable to those of TPWAs, make them ideal candidates for reading out single qubits, or several qubits in a time-multiplexed manner. For these reasons, the preference between amplifier topologies – standing-wave or traveling-wave – for future qubit readout in large quantum processors remains to be answered.

Beyond improving the primary amplifier figures of merit – gain, bandwidth, saturation power, and added noise – parametric amplifier research is now focusing on additional figures of merit that are increasingly important for qubit readout and quantum communication. These include the development of parametric amplifiers with minimized back-action onto the signal source (discussed below), the improvement of squeezing characteristics [76, 89, 90], and the development of magnetic-field-resilient parametric amplifiers. The latter have applications such

as reading out spin qubits [20, 175, 176]. Resonator-based parametric amplifiers utilizing the kinetic inductance of various material classes and architectures were demonstrated to withstand magnetic fields of 0.5–2 T [177, 178, 48, 47], or even up to 6 T [179]. Similar resilience of a TWPA has yet to be demonstrated.

Suggestions for further optimization and future investigations for TWPAs (and parametric amplifiers in general) are provided in the following.

5.2.1 Reducing the system noise

Approaching the quantum limit of noise is a major challenge in the development of amplification chains with a TWPA as the first amplifier. Possible improvements tackle various aspects of the TWPA itself as well as other elements of the amplification chain. The most straight-forward improvement is to achieve sufficiently high TWPA gain to overcome the noise contributions of subsequent amplifier stages. Increasing the gain of our TWPA concept is possible via increasing the pump frequency, nonlinearity, or device length, slightly increasing the dispersion to further suppress unwanted processes, and via decreasing loss and reflections. The latter limit the achievable gain by the onset of parametric oscillations, ultimately depleting the pump.

Reducing the added noise of the TWPA primarily requires reducing its loss. In our device, loss is dominated by the dielectric of the parallel-plate capacitors, PECVD-SiO_x, known to exhibit loss tangents of ca. 3×10^{-3} [84]. It is therefore an important task to replace it by a low-loss dielectric, such as hydrogenated amorphous silicon, or silicon nitride. For example, for parallel-plate capacitors using hydrogenated amorphous silicon as dielectric, deposited by PECVD, loss tangents on the order of 10^{-5} in the single-photon regime were reported [180]. Recently, a TWPA based on that dielectric has shown added noise approaching the quantum limit [161]. As an alternative, parallel-plate capacitors could be replaced by planar capacitors, circumventing the need for a dielectric layer, or by vacuum-gap capacitors [145]. In our design, parallel-plate capacitors also connect the ground planes of the coplanar waveguide to prevent slotline modes. When using planar capacitors instead of parallel-plate capacitors, airbridges can be used to fulfill this purpose.

Another source of excess noise are the sidebands of the signal and the idler, possibly dominating the excess noise in TWPAs with low loss. As discussed in detail in this work, sidebands can be suppressed by dispersion-engineering techniques to some extent. Stronger suppression is possible, however, via the Floquet-mode TWPA concept proposed by Peng et al. [98]. This concept modifies the effective

pump strength along the TWPA circuit in a periodic manner, effectively decoupling the signal and idler modes from their sidebands.

Apart from the added noise of the TWPA, additional contributions to system noise come from the lossy elements before (and after) the TWPA. Therefore, on-chip integration of superconducting passive microwave devices, such as bias tees to supply bias current, or the directional coupler supplying the pump, present a promising task. The same purpose would be fulfilled by the implementation of the flux-pumped TWPA concept, proposed by Zorin [140], in which the signal and pump waves propagate on separate transmission lines. This concept does not require a directional coupler for the pump, nor bias tees for flux biasing. Lastly, reducing the backaction of the TWPA on the signal source to a sufficient extent potentially eliminates the need for lossy isolators between the TWPA and the signal source, further decreasing system noise.

5.2.2 Eliminating gain ripples and backaction: Apodization and backward isolation

Strategies to eliminate gain ripples and backaction can be divided into two groups: First, the reduction of reflections, and second, methods for the isolation or deamplification of backward-traveling waves. The former emphasizes the importance of proper microwave design for broadband impedance matching, including improved packaging [181, 182], for example, by implementing circuits for compensating bond-wire inductances, or using flip-chip bonding instead [183]. Regarding TWPAs with periodic circuit parameter variation (e.g., mPCV), stopband sidelobes have been identified in this work as a limit on the achievable impedance match. To unlock the full potential of the in-situ flux tunability of the impedance of a SQUID-based TWPA, these sidelobes have to be suppressed via techniques such as apodization (Sec. 3.3.4). Our near-term goal is the implementation of an rf-SQUID-based TWPA with dispersion engineering via apodized PCV or mPCV.

Most promising in various aspects, however, is the implementation of an amplifier which amplifies in forward direction and deamplifies (isolates) in reverse direction. A number of concepts and experimental implementations of ferrite-free superconducting microwave isolators and circulators exist, either resonator-based [184, 185, 186, 187] or utilizing traveling-wave architectures with parametric frequency conversion [122] or parametric mode conversion in coupled transmission lines [188, 121]. Concepts to combine isolation (or circulation) with parametric amplification include reservoir engineering [189, 190], frequency conversion [191], mode interference between two JPAs connected via a hybrid coupler [192], and utilizing superconducting switches to isolate the parametric amplifier from the

signal source [193]. In most of these concepts and realizations, isolation and amplification are rather narrowband. However, concepts have been proposed to achieve wideband amplification and reverse isolation by arranging two (or a power of two) TWPAs in a balanced-amplifier topology, connected via diplexers and hybrid couplers [194, 195]. Moreover, two successful implementations of TWPAs with wideband forward amplification and reverse isolation based on frequency conversion were recently demonstrated by research groups at CNRS Grenoble (France) and NIST Boulder (USA), but have not yet been published [196, 197].

5.2.3 Final remarks

Despite the remarkable progress in the last years, there remains room for improvement particularly in the system noise of amplification chains with a TWPA as the first amplifier, and in the backaction on the signal source. Nevertheless, the development of TWPAs remains an exciting field, and next steps for possible improvements have been identified. The models, simulation techniques, and experimental protocols presented in this thesis pave the way for further optimization and engineering of TWPAs. Besides applications like the frequency-multiplexed readout of qubits, promising for further advancing the scale-up of quantum computers, these devices are extremely interesting for fundamental research in quantum communication and quantum sensing due to their potential applications beyond amplification, such as creating wideband nonclassical microwave radiation [70, 76, 89, 90].

Appendix

A.1 Derivation of the model of the nonideal rf-SQUID

A.1.1 Nonideal rf-SQUID with parasitic inductance

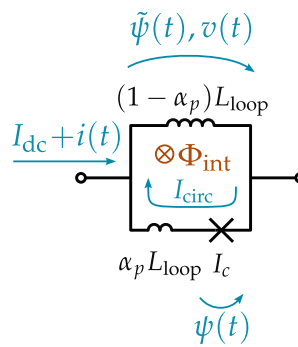


Fig. A.1.: Nonideal rf-SQUID with parasitic inductance in the Josephson junction branch.

To derive the nonlinear current-phase relation $i(\tilde{\phi})$ of a nonideal rf-SQUID with a parasitic inductance $\alpha_p L_{\text{loop}}$, we first define the total phase drops $\psi = \phi_{\text{dc}} + \phi(t)$ and $\tilde{\psi} = \tilde{\phi}_{\text{dc}} + \tilde{\phi}(t)$ (Fig. A.1). Here, ψ corresponds to the gauge-invariant phase difference across the Josephson junction, and hence to the internal magnetic flux (Eq. 2.16), $\psi = -\varphi = \Phi_{\text{int}}/\varphi_0$, while $\tilde{\psi}$ refers to the phase difference across the nodes of the rf-SQUID, such that $v(t) = \varphi_0 \partial \tilde{\psi} / \partial t$. With these definitions, Eq. 2.32 reads

$$I_{\text{dc}} + i(t) = \frac{\varphi_0}{(1 - \alpha_p)L_{\text{loop}}} [\psi + \beta_L \sin(\psi)]. \quad (\text{A.1})$$

The circulating current $I_{\text{circ}} = I_c \sin \varphi$ flows through both the Josephson junction and the parasitic inductance,

$$I_{\text{circ}} = -I_c \sin \psi = \varphi_0 \frac{\psi - \tilde{\psi}}{\alpha_p L_{\text{loop}}}, \quad (\text{A.2})$$

leading to the relation between $\tilde{\psi}$ and ψ ,

$$\tilde{\psi} = \psi + \alpha_p \beta_L \sin \psi, \quad (\text{A.3})$$

which is an implicit equation in ψ . Therefore, we can only partially substitute ψ by $\tilde{\psi}$ in Eq. A.1,

$$\begin{aligned} I_{dc} + i(t) &= \frac{\varphi_0}{(1 - \alpha_p)L_{loop}} [\tilde{\psi} - \alpha_p \beta_L \sin \psi + \beta_L \sin \psi] \\ &= \frac{\varphi_0}{(1 - \alpha_p)L_{loop}} [\tilde{\psi} + \beta_L(1 - \alpha_p) \sin \psi]. \end{aligned} \quad (A.4)$$

Next, we expand the term $\sin(\psi(\tilde{\psi}))$ to a Taylor series in $\tilde{\psi}$, using the implicit derivative of $\psi(\tilde{\psi})$, which is

$$\frac{\partial \psi}{\partial \tilde{\psi}} = \left(\frac{\partial \tilde{\psi}}{\partial \psi} \right)^{-1} = \frac{1}{1 + \alpha_p \beta_L \cos \psi}. \quad (A.5)$$

With this, the first three derivatives, appearing in the Taylor expansion of $y(\tilde{\psi}) = \sin(\psi(\tilde{\psi}))$, can be calculated iteratively,

$$\frac{dy(\tilde{\psi})}{d\tilde{\psi}} = \frac{\partial y}{\partial \psi} \frac{\partial \psi}{\partial \tilde{\psi}} = \frac{\cos \psi}{1 + \alpha_p \beta_L \cos \psi} =: y' \quad (A.6)$$

$$\frac{d^2 y(\tilde{\psi})}{d\tilde{\psi}^2} = \frac{\partial y'}{\partial \psi} \frac{\partial \psi}{\partial \tilde{\psi}} = -\frac{\sin \psi}{(1 + \alpha_p \beta_L \cos \psi)^3} =: y'' \quad (A.7)$$

$$\frac{d^3 y(\tilde{\psi})}{d\tilde{\psi}^3} = \frac{\partial y''}{\partial \psi} \frac{\partial \psi}{\partial \tilde{\psi}} = -\frac{(1 + \alpha_p \beta_L \cos \psi) \cos \psi + 3\alpha_p \beta_L \sin^2 \psi}{(1 + \alpha_p \beta_L \cos \psi)^5} =: y'''. \quad (A.8)$$

We expand $\sin \tilde{\psi}$ as a function of $\tilde{\phi}(t)$ in the vicinity of $\tilde{\psi} = \tilde{\phi}_{dc}$, equivalent to $\psi = \phi_{dc}$, and find

$$\sin \psi = \sin \phi_{dc} + y' \Big|_{\phi_{dc}} \tilde{\phi} + \frac{1}{2} y'' \Big|_{\phi_{dc}} \tilde{\phi}^2 + \frac{1}{6} y''' \Big|_{\phi_{dc}} \tilde{\phi}^3 + \dots, \quad (A.9)$$

which can be inserted into Eq. A.4,

$$\begin{aligned} I_{dc} + i(t) &= \frac{\varphi_0}{(1 - \alpha_p)L_{loop}} \left[\phi_{dc} + \beta_L \sin \phi_{dc} \right. \\ &\quad + \left(1 + \beta_L(1 - \alpha_p) \frac{\cos \psi}{1 + \alpha_p \beta_L \cos \psi} \right) \tilde{\phi} \\ &\quad - \frac{\beta_L(1 - \alpha_p)}{2} \frac{\sin \psi}{(1 + \alpha_p \beta_L \cos \psi)^3} \tilde{\phi}^2 \\ &\quad \left. - \frac{\beta_L(1 - \alpha_p)}{6} \frac{(1 + \alpha_p \beta_L \cos \psi) \cos \psi + 3\alpha_p \beta_L \sin^2 \psi}{(1 + \alpha_p \beta_L \cos \psi)^5} \tilde{\phi}^3 + \dots \right]. \end{aligned} \quad (A.10)$$

The constant terms on the right-hand side correspond to the current bias I_{dc} , and the term linear in $\tilde{\phi}$ contains the small-signal inductance $L_{SQ}(\phi_{dc})$ (Eq. 2.31). Finally,

after eliminating the constant terms and normalizing the nonlinear terms by the prefactor of the linear term we obtain Eqs. 2.33–2.35 of Sec. 2.1.3,

$$i(\tilde{\phi}) = \frac{\varphi_0}{L_{\text{SQ}}(\phi_{\text{dc}})} [\tilde{\phi} - \beta \tilde{\phi}^2 - \gamma \tilde{\phi}^3 + \dots], \quad (\text{A.11})$$

$$\beta = \frac{\beta_L(1 - \alpha_p)}{2} \frac{\sin \phi_{\text{dc}}}{(1 + \beta_L \cos \phi_{\text{dc}})(1 + \alpha_p \beta_L \cos \phi_{\text{dc}})^2}, \quad (\text{A.12})$$

$$\gamma = \frac{\beta_L(1 - \alpha_p)}{6} \frac{(1 + \alpha_p \beta_L \cos \phi_{\text{dc}}) \cos \phi_{\text{dc}} + 3\alpha_p \beta_L \sin^2 \phi_{\text{dc}}}{(1 + \beta_L \cos \phi_{\text{dc}})(1 + \alpha_p \beta_L \cos \phi_{\text{dc}})^4}. \quad (\text{A.13})$$

A.1.2 Quasi-one-junction SQUID

To investigate the properties of a quasi-one-junction SQUID (Sec. 2.1.3), we first consider the general case of a SQUID with two Josephson junctions in one branch, and the linear inductance L_m in the other branch (Fig. A.2). The Josephson junctions have critical currents $I_{c,1}$ and $I_{c,2}$, and gauge-invariant phase differences φ_1 and φ_2 , respectively. Adapting the result of a one-junction SQUID, Eq. 2.16, with $\varphi = \varphi_1 + \varphi_2$, we get an expression for the internal magnetic flux,

$$\varphi_1 + \varphi_2 = -\frac{2\pi}{\Phi_0} \Phi_{\text{int}}. \quad (\text{A.14})$$

The circulating current flows through both Josephson junctions,

$$I_{\text{circ}} = I_{c,1} \sin \varphi_1 = I_{c,2} \sin \varphi_2, \quad (\text{A.15})$$

so the phase difference of one junction can be expressed by that of the other,

$$\varphi_2 = \arcsin(r \sin \varphi_1), \quad (\text{A.16})$$

where the ratio $r = I_{c,1}/I_{c,2}$ was introduced. Summing up the contributions to the internal magnetic flux,

$$\begin{aligned} \Phi_{\text{int}} &= \Phi_{\text{ext}} + L_m I_{\text{circ}} \\ &= \Phi_{\text{ext}} + L_m I_{c,1} \sin \varphi_1, \end{aligned} \quad (\text{A.17})$$

and substituting Eqs. A.14 and A.16, yields the relation

$$\frac{2\pi\Phi_{\text{ext}}}{\Phi_0} = -\varphi_1 - \arcsin(r \sin \varphi_1) - \frac{2\pi L_m I_{c,1}}{\Phi_0} \sin \varphi_1, \quad (\text{A.18})$$

which can be solved self-consistently together with Eq. A.16 for a given external magnetic flux, generated e.g. by an injected current.

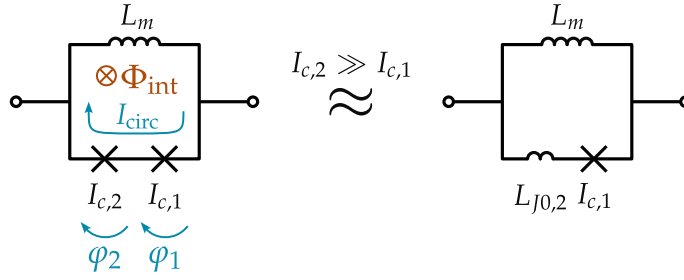


Fig. A.2.: A quasi-one-junction SQUID and its approximation for $I_{c,2} \gg I_{c,1}$.

However, in a quasi-one-junction SQUID, one junction is much larger than the other, e.g., $I_{c,2} \gg I_{c,1}$, or equivalently, $r \ll 1$. This justifies the approximation $\varphi_2 = \arcsin(r \sin \varphi_1) \approx r \sin \varphi_1$, leading to

$$\begin{aligned} \frac{2\pi\Phi_{\text{ext}}}{\Phi_0} &\approx -\varphi_1 - r \sin \varphi_1 - \frac{2\pi L_m I_{c,1}}{\Phi_0} \sin \varphi_1 \\ &= -\varphi_1 - \left(\frac{2\pi L_m I_{c,1}}{\Phi_0} + r \right) \sin \varphi_1, \end{aligned} \quad (\text{A.19})$$

which has a similar form as Eq. 2.18, so it is practical to introduce the screening parameter of the quasi-one-junction SQUID,

$$\beta_L^{\text{QOJS}} = \frac{2\pi L_m I_{c,1}}{\Phi_0} + r = \frac{2\pi L_m I_{c,1}}{\Phi_0} + \frac{I_{c,1}}{I_{c,2}}, \quad (\text{A.20})$$

which differs from that of a one-junction SQUID by the additional term $I_{c,1}/I_{c,2}$. For any φ_1 , the phase drop on the more transparent junction with $I_{c,2}$ is small,

$$|\varphi_2| \approx r |\sin \varphi_1| \ll 1. \quad (\text{A.21})$$

As a consequence, its small-signal inductance

$$L_{J,2}(\varphi_2) = \frac{\Phi_0}{2\pi I_{c,2}} \frac{1}{\cos \varphi_2} \quad (\text{A.22})$$

can be approximated using $\cos(\arcsin x) = \sqrt{1-x^2} \approx (1+x^2/2)^{-1}$, valid for $|x| \ll 1$, and $x = r \sin \varphi_1$, yielding

$$L_{J,2}(\varphi_2) \approx L_{J0,2} [1 + (r/2)^2 (1 - \cos 2\varphi_1)] \quad (\text{A.23})$$

$$\approx L_{J0,2} = \frac{\Phi_0}{2\pi I_{c,2}}. \quad (\text{A.24})$$

Since the second term in the bracket scales with r^2 , the latter approximation holds even for a ratio r that is not much smaller than 1. For example, for $r = 1/7$, the

error is already smaller than 1% for any value of φ_1 . With this approximation, the small-signal inductance of the quasi-one-junction SQUID is

$$\begin{aligned} L_{\text{QOJS}}(\phi_{\text{dc}}) &= \left(\frac{1}{L_m} + \frac{1}{L_{J0,2} + L_{J0,1} / \cos \phi_{\text{dc}}} \right)^{-1} \\ &= L_m \frac{1 + r \cos \phi_{\text{dc}}}{1 + \beta_L^{\text{QOJS}} \cos \phi_{\text{dc}}}, \end{aligned} \quad (\text{A.25})$$

where the constant phase ϕ_{dc} is given by $\Phi_{\text{ext,dc}} / \varphi_0 = \phi_{\text{dc}} + \beta_L^{\text{QOJS}} \cos \phi_{\text{dc}}$. Moreover, $|\varphi_2| \ll 1$ holds for any applied magnetic flux or injected current, and therefore the large junction behaves as a linear inductance $L_{J0,2}$, even for large-signal excitations on the SQUID.

For these reasons, the quasi-one-junction SQUID can be approximated by a nonideal rf-SQUID with a parasitic inductance $L_p = L_{J0,2}$ in the junction branch, as shown in Fig. A.2. Indeed, the screening parameter, Eq. A.20, is identical to

$$\beta_L^{\text{QOJS}} = \frac{2\pi(L_m + L_{J0,2})I_{c,1}}{\Phi_0} = \beta_L, \quad (\text{A.26})$$

which is similar to that of an rf-SQUID with loop inductance $L_{\text{loop}}^{\text{QOJS}} = L_m + L_{J0,2}$. By introducing the nonideality parameter

$$\alpha_p^{\text{QOJS}} = \frac{L_{J0,2}}{L_m + L_{J0,2}}, \quad (\text{A.27})$$

all results of Appendix A.1.1 apply to the quasi-one-junction SQUID.

A.1.3 Nonideal rf-SQUID with lead inductance

Finally, we extend the model of the nonideal rf-SQUID of Sec. A.1.1 by including the inductance of the feedlines connected to the SQUID, L_f . The total phase drop $\hat{\psi} = \hat{\phi}_{\text{dc}} + \hat{\phi}(t)$ across the nodes determines the voltage $v(t)$ (Fig. A.3). The phase drop on L_f ,

$$\hat{\psi} - \tilde{\psi} = L_f(I_{\text{dc}} + i(t)) / \varphi_0, \quad (\text{A.28})$$

in conjunction with Eqs. A.1 and A.3, and $L_m = (1 - \alpha_p)L_{\text{loop}}$, establishes the relation

$$\begin{aligned} \hat{\psi} &= \left(1 + \frac{L_f}{L_m} \right) \psi + \beta_L \left(\alpha_p + \frac{L_f}{L_m} \right) \sin \psi \\ &= \frac{1}{p} (\psi + \alpha_f \beta_L \sin \psi). \end{aligned} \quad (\text{A.29})$$

Here, the parameter p , defined as

$$p = \frac{L_m}{L_f + L_m} \quad (\text{A.30})$$

can be interpreted as the inductance participation ratio at $\phi_{\text{dc}} = \pi/2$, indicating the proportion of the SQUID inductance to the total inductance, and

$$\alpha_f = \frac{\alpha_p L_m + L_f}{L_m + L_f} \quad (\text{A.31})$$

denotes a modified nonideality coefficient. The first derivative of Eq. A.29 is

$$\frac{\partial \psi}{\partial \hat{\psi}} = \left(\frac{\partial \hat{\psi}}{\partial \psi} \right)^{-1} = \frac{p}{1 + \alpha_f \beta_L \cos \psi}. \quad (\text{A.32})$$

Next, we express the current-phase relation of the SQUID, Eq. A.1, as a function of $\hat{\psi}$,

$$I_{\text{dc}} + i(t) = \frac{\varphi_0}{L_m} [p\hat{\psi} + \beta_L(1 - \alpha_f) \sin(\psi(\hat{\psi}))]. \quad (\text{A.33})$$

This relation is of the same form as Eq. A.4. Again, the remaining term $\sin(\psi(\hat{\psi}))$ can be expanded into a Taylor series in $\hat{\psi}$ by using the implicit derivative of the transcendental Eq. A.29. Performing the same steps as between Eqs. A.6 and A.10, we obtain the nonlinear current-phase relation

$$i(\hat{\phi}) = \frac{\varphi_0}{L_{\text{SQ}}(\phi_{\text{dc}})} [\hat{\phi} - \beta \hat{\phi}^2 - \gamma \hat{\phi}^3 + \dots], \quad (\text{A.34})$$

with the coefficients of second-order and third-order nonlinearity,

$$\beta = \frac{p\beta_L(1 - \alpha_f)}{2} \frac{\sin \phi_{\text{dc}}}{(1 + \beta_L \cos \phi_{\text{dc}})(1 + \alpha_f \beta_L \cos \phi_{\text{dc}})^2}, \quad (\text{A.35})$$

$$\gamma = \frac{p^2 \beta_L(1 - \alpha_f)}{6} \frac{(1 + \alpha_f \beta_L \cos \phi_{\text{dc}}) \cos \phi_{\text{dc}} + 3\alpha_f \beta_L \sin^2 \phi_{\text{dc}}}{(1 + \beta_L \cos \phi_{\text{dc}})(1 + \alpha_f \beta_L \cos \phi_{\text{dc}})^4}, \quad (\text{A.36})$$

and the small-signal inductance

$$L_{\text{SQ}} = (L_f + L_m) \frac{1 + \alpha_f \beta_L \cos \phi_{\text{dc}}}{1 + \beta_L \cos \phi_{\text{dc}}} \quad (\text{A.37})$$

$$= L_f + L_{\text{loop}} \frac{(1 - \alpha_p)(1 + \alpha_p \beta_L \cos \phi_{\text{dc}})}{1 + \beta_L \cos \phi_{\text{dc}}}. \quad (\text{A.38})$$

Due to the parasitic inductances L_p and L_f , the Josephson junction capacitance C_J (Fig. A.3) has a slightly different effect on the impedance of the inductive element

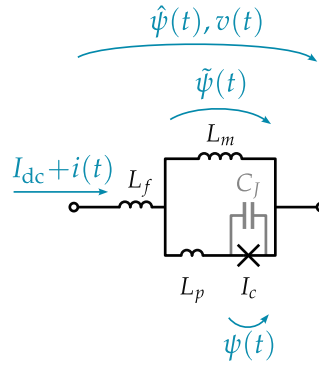


Fig. A.3.: A nonideal rf-SQUID with parasitic inductance L_p and lead inductance L_f .

compared to an ideal rf-SQUID. The poles and zeros of the reactance $\text{Im}\{Z_{\text{SQ}}(\omega)\}$ of the inductive element relate to different resonances. In the case of a nonideal rf-SQUID with L_p and L_f , this impedance exhibits one pole and one zero (apart from the trivial zero at $\omega = 0$). The pole corresponds to the plasma frequency,

$$\omega_J = \sqrt{\frac{1 + \beta_L \cos \phi_{\text{dc}}}{C_J(L_m + L_p)}}. \quad (\text{A.39})$$

which is an important quantity determining the dispersion in TWPAs (see Sec. 2.3). At this frequency, $\text{Im}\{Z_{\text{SQ}}\}$ diverges due to a resonance of the SQUID loop with the capacitance C_J , independent of L_f . For frequencies above ω_J , the SQUID loop exhibits capacitive behaviour. When its reactance is equal to ωL_f , a series resonance occurs at a frequency $\omega_{J,2} > \omega_J$, resulting in $\text{Im}\{Z_{\text{SQ}}(\omega_{J,2})\} = 0$. Above this second resonance frequency, the SQUID exhibits inductive behaviour again, because ωL_f dominates the impedance.

In a transmission line consisting of nonideal rf-SQUIDs, the parasitic inductances L_p and L_f alter the dispersion relation and the cutoff frequency, Eqs. 2.47 and 2.49. If the plasma frequency ω_J is substantially smaller than ω_0 , as it is the case in the experimental implementation presented in this work (c.f. circuit parameters in Tab. 4.4), the cutoff frequency is rather independent of L_f and can be approximated to $\omega_c \approx 2/\sqrt{(L_m + L_p)(C_0 + 4C_J)}$ (compare to Eq. 2.49). Note that the series resonance of C_J with L_f , occurring at a higher frequency than ω_c , results in a second and a third cutoff frequency, which delimit a second passband.

A.2 Fabrication process monitoring

A.2.1 Josephson junctions

To investigate the properties of the Josephson junctions used in the TWPA circuits, we measured co-fabricated Josephson-junction test structures at 4.2 K and at room temperature. These test structures consist either of single Josephson junctions of the same size and geometry as those in the TWPA circuits, or of series connections of two junctions with $289 \mu\text{m}^2$ each, see insets in Fig. A.4. The former are embedded in surrounding structures similar to the rf-SQUIDs but without an inductor in parallel. In this way, these test structures are fabricated under conditions comparable to those of rf-SQUIDs in TWPA arrays.

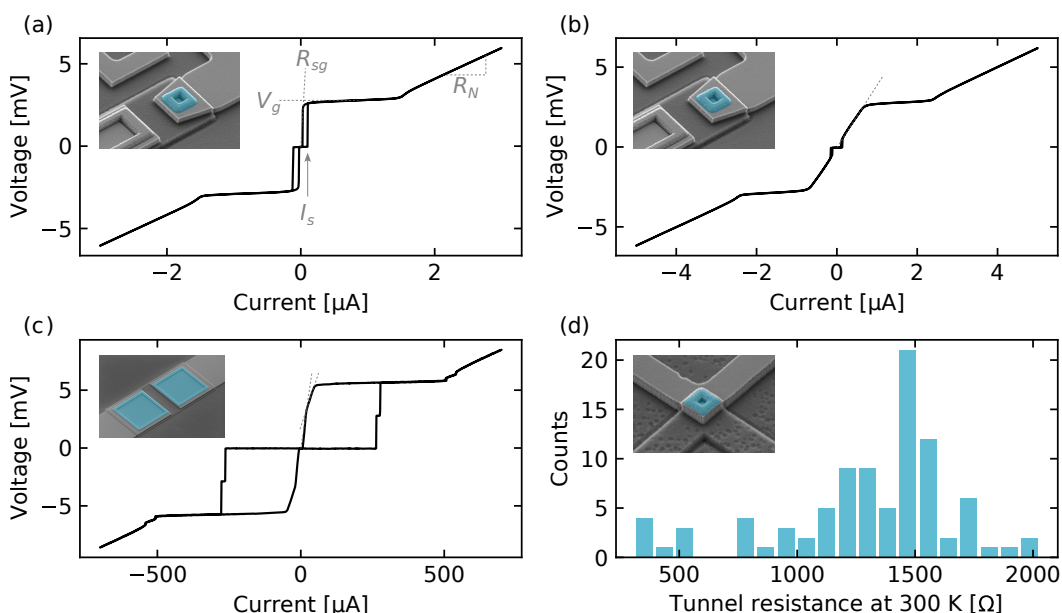


Fig. A.4.: Measurements on Josephson junction test structures. **(a)-(c)** Current-voltage characteristics (IVC) of different test structures of the same wafer measured at 4.2 K. The structures of Panels **(a),(b)** are similar to the rf-SQUIDs but are not shunted by an inductor. Although the Josephson junctions are nominally identical, their different R_{sg}/R_N ratios of 37 and 3.3, respectively, indicate very different barrier qualities. The critical currents of both junctions with sizes $1 \times 1 \mu\text{m}^2$ are strongly suppressed due to the small value of I_c . The 14 times larger junctions used as vias remain superconducting in the whole current range of the shown IVCs. **(c)** IVC of a series connection of two nominally identical Josephson junctions with the sizes $17 \times 17 \mu\text{m}^2$, showing slightly different I_c and R_{sg} values. **(d)** Histogram of tunnel resistances of 91 Josephson junctions with sizes $1 \times 1 \mu\text{m}^2$ measured at room temperature. Blue regions in the false-color insets indicate Josephson junctions. Scanning electron micrographs by Jan Blohm and Victor Gaydamachenko.

Critical current density

The average critical-current density J_c of the wafer is obtained in two different ways from current-voltage characteristics measured at 4.2 K. The critical currents of the small junctions with a size of $A = 1 \mu\text{m}^2$ could not be measured directly, because their switching current $I_{\text{sw}} \ll I_c$ is affected by current noise from the measurement setup (Fig. A.4a,b). Instead, the normal resistances R_N and the gap voltages V_g of the junctions are extracted from current-voltage characteristics as schematized in Fig. A.4a. Using the Ambegaokar-Baratoff relation [198], the critical current at $T = 4.2 \text{ K}$ is

$$I_c = g \frac{\pi}{2e} \frac{\Delta(T)}{R_N} \tanh\left(\frac{\Delta(T)}{2k_B T}\right), \quad (\text{A.40})$$

with the elementary charge e , the Boltzmann constant k_B , and the superconducting energy gap of the Nb electrodes, $\Delta = eV_g/2$, assuming that both electrodes have the same energy gap. The coefficient $g = 0.79$ was previously determined in well-filtered measurements on junctions large enough that their critical currents are assumed to be not suppressed. It compares to literature values of 0.75–0.9 [78, 199] for Nb/Al-AlO_x/Nb tunnel junctions and reflects nonidealities such as the proximity effect of the 10 nm thin Al on top of the Nb base electrode, or strong-coupling effects in Nb [78]. The resulting average critical-current density $J_c = I_c/A = 0.992 \mu\text{A}/\mu\text{m}^2$ is slightly lower than the planned $1.3 \mu\text{A}/\mu\text{m}^2$. The critical current of the large junctions with $A = 289 \mu\text{m}^2$ can be measured directly, as $I_{\text{sw}} \approx I_c$, see Fig. A.4c. The average critical-current density of $J_c = 0.88 \mu\text{A}/\mu\text{m}^2$ is lower than that obtained by the latter method, indicating that the critical current in these junctions is still partially suppressed.

Specific capacitance

The specific capacitance of the Josephson junctions can be estimated from J_c using the approximate expression adapted from Ref. [200],

$$c_j \approx \frac{1}{0.0243 - 0.0043 \log_{10}(J_c/\mu\text{A}/\mu\text{m}^2)} \text{ fF}/\mu\text{m}^2, \quad (\text{A.41})$$

yielding $C_J \approx 41 \text{ fF}$ for a $1 \mu\text{m}^2$ sized junction. Taking into account a few femtofarads of capacitance of the connecting electrodes, we take $C_J = 45 \text{ fF}$ for the TWPA circuit model.

Subgap resistance and tunnel barrier quality

The subgap resistance of a Josephson junction plays a role as a source of loss in TWPAs, further analyzed in Sec. 4.2.1, and is a nonlinear function of the voltage.

In an rf-SQUID, the junction is operated in the zero-voltage state, with a subgap resistance often referred to as the intrinsic subgap resistance [201]. It is not possible to directly measure the intrinsic subgap resistance because the current does not decrease monotonically to zero in the subgap region $V < V_g$. Instead, the junction switches back to the superconducting state at the so-called re-trapping current [202]. We define the subgap resistance as the maximum resistance in the voltage range below V_g . This method gives a lower bound on R_{sg} , while the intrinsic subgap resistance can be much larger [203]. To measure the intrinsic subgap resistance is much more elaborate [201, 204, 203] and is beyond the scope of this work.

The quality of the tunnel barrier is evaluated via the ratio of the subgap resistance and the normal-state resistance, R_{sg}/R_N , which is independent of the junction geometry. Barriers with $R_{sg}/R_N > 10$ at 4.2 K are typically considered as of high quality. On the present wafer, this ratio ranged between 2 and 37, showing sporadically very leaky barriers (Fig. A.4), which we attribute to unwanted resistive paths parallel to the junctions. Their origin is not yet understood. On 14 previous wafers, measured over the course of four years by Victor Gaydamachenko, Marat Khabipov, and me, the ratio R_{sg}/R_N was 39 ± 7 , indicating consistently high quality. The given uncertainty is one standard deviation for the ensemble of 115 measured junctions. The same applies to the gap voltage, which indicates the quality of the Nb film. The gap voltage on the present wafer is (2.83 ± 0.02) mV, which compares to (2.90 ± 0.04) mV on 14 previous wafers.

Critical current spread

Additionally to the measurements at 4.2 K, the tunnel resistances of 112 single Josephson junctions with a size of $1 \mu\text{m}^2$ across the wafer were measured in a four-point scheme at room temperature using an automated step prober before the wafer was diced into chips. Since the tunnel resistance at room temperature is inversely proportional to the critical current of a junction [199], the distribution of the critical currents across the wafer can be obtained with this method. The resulting on-wafer standard deviation of the tunnel resistances is 29%, which compares to 28% and 23% measured at 4.2 K for Josephson junctions of the sizes $1 \mu\text{m}^2$ and $289 \mu\text{m}^2$, respectively. This spread is huge in comparison to 1–14% on previous wafers, with a mean spread of 5%. A histogram of the data is shown in Fig. A.4d. The histogram is asymmetric towards lower tunnel resistance values, probably having the same cause as the sporadic barrier leakage. It is noteworthy here that although the critical-current spread of this wafer is huge, the devices still work well. This gives evidence to the findings of Chapter 3, where it was predicted that rf-SQUID-based 3WM TWPAs can tolerate high critical-current spread.

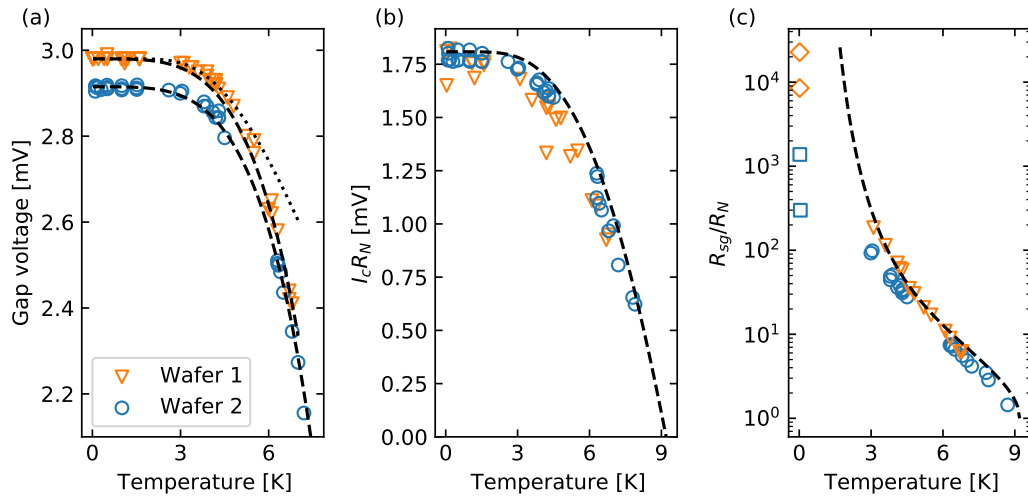


Fig. A.5.: Temperature dependences in Josephson junctions from two different wafers. **(a)** Dependence of the gap voltage on the temperature with two different approximations of $\Delta(T)$. The dashed lines are approximations using Eq. A.43, while the dotted line is given by Eq. A.42. Parameters of the approximations are $2\Delta_0 = 2.98$ meV (2.92 meV) for Wafer 1 (Wafer 2), and $T_c = 9.2$ K. The junctions of the two wafers show slightly different gap voltages, indicating varying qualities of the Nb electrodes. **(b)** The $I_c R_N$ -product stays slightly below the prediction of the Ambegaokar-Baratoff relation, in which Eq. A.43 and $g = 0.79$ were used. **(c)** The subgap resistance follows qualitatively the trend $\propto \exp(\Delta/k_B T)$ given by thermally excited quasiparticles (dashed line), and increases by orders of magnitude between 4.2 K and 40 mK. Triangles and circles denote current-biased measurements, while diamonds and squares denote voltage-biased measurements, giving higher resolution for high subgap resistances. The normal resistance R_N was found temperature-independent in the given temperature range.

Temperature dependence

To examine the temperature dependences of the properties of the Josephson junctions, we measured the current-voltage characteristics of Josephson junctions on two different wafers at different temperatures between ca. 20 mK and 9.3 K in a dilution refrigerator (Fig. A.5). The wafers are different from those of the samples presented in Chapter 4. The gap voltage, the $I_c R_N$ product, and R_{sg}/R_N , all being independent of the junction area and critical-current density, decrease with increasing temperature.

Measured gap voltages agree well with $\Delta(T)$ calculated using approximations of the BCS theory (after Bardeen, Cooper, Schrieffer [205]), although Nb does not strictly follow the BCS theory in its weakly-coupled limit [206]. While the approximate expression [207]

$$\Delta(T) \approx \Delta_0 \exp \left[-\sqrt{\frac{2\pi k_B T}{\Delta_0}} \exp \left(-\frac{\Delta_0}{k_B T} \right) \right] \quad (\text{A.42})$$

fits the data better for $T < T_c/2$, where T_c denotes the critical temperature of the superconductor, the approximation [208]

$$\Delta(T) \approx \Delta_0 \sqrt{\cos\left(\frac{\pi}{2} \left(\frac{T}{T_c}\right)^2\right)}, \quad (\text{A.43})$$

is suitable for the whole temperature range $0 \dots T_c$ and is used in the remainder of the text.

The temperature dependences of the critical currents follow the Ambegaokar-Baratoff relation, Eq. A.40. According to Eq. A.40, I_c reduces by ca. 6% between 4.2 K and 20 mK. Systematically reduced critical currents for $T > 1$ K are typically explained by the proximity effect of the Al layer on top of the Nb base electrode [209]. Wafer 1 has a J_c half that of Wafer 2, so the switching currents of its junctions were more severely suppressed by noise, appearing as reduced critical currents.

The subgap resistance has a complex temperature dependence, involving different mechanisms including quasiparticle-pair-interference currents and quantum retrapping [202]. For high-quality junctions, R_{sg} is dominated by thermally excited quasiparticle currents [201], resulting in $R_{sg} \propto \exp(\Delta/k_B T)$ for $k_B T \ll \Delta$ [202]. The data shown in Fig. A.5c qualitatively follows this trend. For temperatures in the millikelvin range, R_{sg} is orders of magnitude higher than at 4.2 K, but is limited by effects other than thermally excited quasiparticles [210, 211]. Because of low resolution of the high-ohmic subgap region in a current-biased measurement, the measurements at ca. 40 mK were performed in a voltage-biased regime (diamonds and squares in Fig. A.5c).

A.2.2 Niobium film

Complex conductivity

In analogy to a normal conductor, where the current density $\vec{J}$ and the electrical field $\vec{E}$ can be related by a real-valued conductivity σ_n , the corresponding relationship in a superconductor can be written as

$$\vec{J} = \sigma \vec{E} = (\sigma_1 - j\sigma_2) \vec{E} \quad (\text{A.44})$$

with a complex conductivity σ [212]. This conductivity can be understood in the light of a two-fluid model, describing the interplay of both single-electron and paired-electron excitations, referred to as quasiparticles and Cooper pairs, respectively. Both conductivities σ_1 and σ_2 are frequency and temperature dependent and are linked to the normal-state conductivity σ_n of the material, typically measured

just above T_c . They can be calculated according to the Mattis-Bardeen theory [213] as follows:

$$\frac{\sigma_1}{\sigma_n} = \frac{2}{\hbar\omega} \int_{\Delta(T)}^{\infty} [f(E) - f(E + \hbar\omega)] g(E) dE + \frac{1}{\hbar\omega} \int_{\Delta(T) - \hbar\omega}^{-\Delta(T)} [1 - 2f(E + \hbar\omega)] g(E) dE, \quad (\text{A.45})$$

$$\frac{\sigma_2}{\sigma_n} = \frac{1}{\hbar\omega} \int_{\{\Delta(T) - \hbar\omega; -\Delta(T)\}}^{\Delta(T)} \frac{[1 - 2f(E + \hbar\omega)] [E^2 + \Delta(T)^2 + \hbar\omega E]}{\sqrt{\Delta(T)^2 - E^2} \sqrt{(E + \hbar\omega)^2 - \Delta(T)^2}} dE, \quad (\text{A.46})$$

with

$$g(E) = \frac{E^2 + \Delta(T)^2 + \hbar\omega E}{\sqrt{E^2 - \Delta(T)^2} \sqrt{(E + \hbar\omega)^2 - \Delta(T)^2}}, \quad (\text{A.47})$$

and the Fermi-Dirac distribution

$$f(\epsilon) = \frac{1}{1 + \exp(\epsilon/k_B T)}. \quad (\text{A.48})$$

The first and the second integral in Eq. A.45 account for thermally excited quasiparticles and the photon-excited quasiparticles, respectively [78]. If $\hbar\omega < 2\Delta$, the second integral in Eq. A.45 is zero, and $\Delta - \hbar\omega$ is applied as the lower boundary of the integral in Eq. A.46. Although the Mattis-Bardeen theory employs the BCS weak-coupling theory, it was found to be in remarkably good agreement with experimental data for Nb, which itself is an anisotropic strong-coupling superconductor [214]. Figure A.6 shows the temperature dependences of σ_1 and σ_2 , calculated by numerically solving Eqs. A.45 and A.46 for a Nb film with $\Delta(T)$ given by Eq. A.43, and for a frequency of 10 GHz.

In some limiting cases the Eqs. A.45 and A.46 can be approximated by simple analytic expressions. Under the conditions $\hbar\omega \ll 2\Delta$ and $k_B T \ll \Delta$, Eq. A.45 approximates to

$$\frac{\sigma_1}{\sigma_n} = \frac{4\Delta(T)}{\hbar\omega} \exp\left(-\frac{\Delta(T)}{k_B T}\right) \sinh\left(\frac{\hbar\omega}{2k_B T}\right) K_0\left(\frac{\hbar\omega}{2k_B T}\right), \quad (\text{A.49})$$

where K_n is the n th-order modified Bessel function of the second kind [215]. For $T < 0.6T_c$ and at 10 GHz, the error of this approximation stays within $\pm 5\%$ when compared to the Mattis-Bardeen theory (Fig A.6). In a low-frequency limit $\hbar\omega \ll 2\Delta$, Eq. A.46 can be replaced by the simple expression [206]

$$\frac{\sigma_2}{\sigma_n} = \frac{\pi\Delta(T)}{\hbar\omega} \tanh \frac{\Delta(T)}{2k_B T}, \quad (\text{A.50})$$

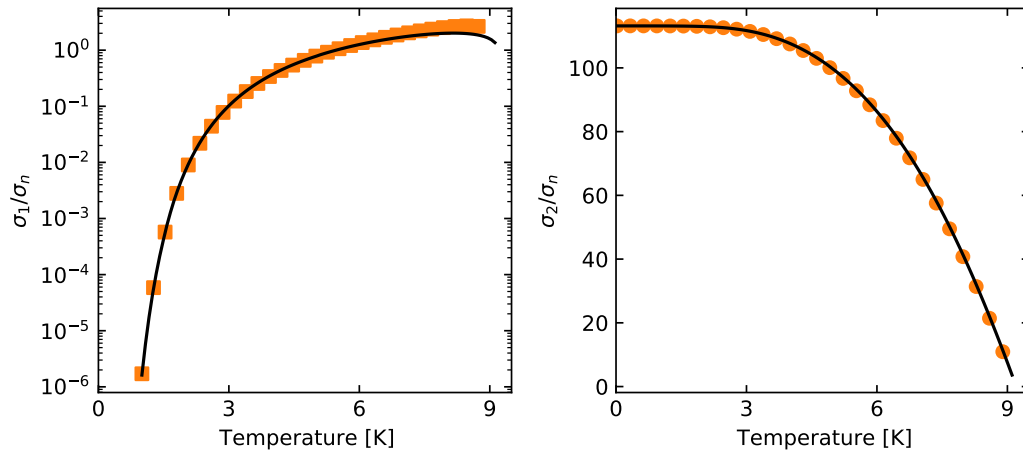


Fig. A.6.: Temperature dependences of the real and the imaginary part of the complex conductivity $\sigma = \sigma_1 - i\sigma_2$ for a frequency of 10 GHz. Solid lines are computed by numerically solving the Mattis-Bardeen theory (Eqs. A.45 and A.46), using the approximation Eq. A.43 for $\Delta(T)$, with $T_c = 9.2$ K and $2\Delta_0 = 2.98$ meV. Squares and circles are the approximations Eqs. A.49 and A.50, respectively.

which is in good agreement with Eq. A.46 in the whole temperature range $0 \dots T_c$, as shown in Fig A.6. From these equations it can be seen that as temperatures decrease, σ_1 falls off as $\exp(-\Delta/k_B T)$ due to the freeze-out of the thermally excited quasiparticles, while σ_2 approaches a constant plateau of $\sigma_2/\sigma_n \rightarrow \pi\Delta_0/\hbar\omega$.

Penetration depth

Magnetic fields are expelled from the interior of the superconductor and penetrate only a thin region beneath the surface, with an effective thickness given by the penetration depth

$$\lambda = \sqrt{\frac{1}{\omega\mu_0\sigma_2}}, \quad (\text{A.51})$$

where μ_0 is the vacuum permittivity. Inside the superconductor, the magnetic field and, consequently, the current density decrease exponentially. For example, in an isolated superconducting strip with width w and thickness d , the current density J varies across the thickness as

$$J(x) = \frac{I}{2w\lambda} \frac{\cosh(x/\lambda)}{\cosh(d/2\lambda)}, \quad (\text{A.52})$$

where $-d/2 \leq x \leq d/2$, and the current I flowing through the strip. Assuming $w \gg d$, we can neglect current density variations across the width of the strip.

The penetration depth has major implications on the characteristics of a superconducting film, such as the surface inductance and the surface resistance, which

will be addressed in detail below. It can be determined by measuring the normal conductivity at a temperature just above T_c . Table A.1 lists experimental data of three samples with long and narrow strips, i.e., large aspect ratios ℓ/w often referred to as number of squares $\square$. Sample D is from the same wafer as Samples A–C. Values are obtained using Eqs. A.50 and A.51 with $2\Delta_0 = 2.9$ meV taken from Josephson-junction measurements (Sec. A.2.1). From 0 K to 4.2 K the penetration depth λ increases by only 3%.

Tab. A.1.: Niobium film characterization.

	Sample D	Sample E	Sample F
Squares ℓ/w	19,218 $\square$	19,218 $\square$	240,123 $\square$
Resistance at $T \approx T_c$	2.49 k Ω	2.70 k Ω	29.2 k Ω
Normal conductivity σ_n	2.57×10^7 S/m	2.37×10^7 S/m	2.74×10^7 S/m
Penetration depth λ ($T = 0$ K)	67 nm	70 nm	65 nm
Surface inductance L_s ($T = 0$ K)	43 fH/ $\square$	45 fH/ $\square$	42 fH/ $\square$

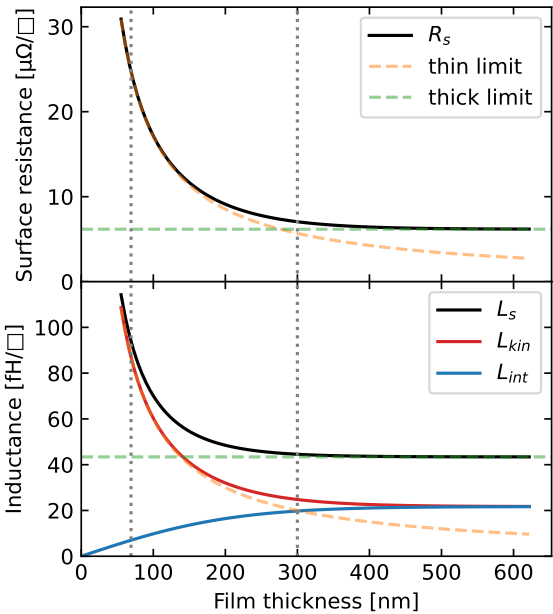


Fig. A.7.: Surface impedance $Z_s = R_s + j\omega L_s$ of a superconducting strip for $f = 10$ GHz and $T = 4.2$ K. The surface inductance has two contributions: the kinetic inductance L_{kin} and the internal magnetic inductance L_{int} . Vertical dotted lines mark the film thickness $d = 300$ nm and the penetration depth $\lambda = 69$ nm.

Contributions to the inductance of a superconducting film

A method to determine the inductance L of a superconducting structure is to relate the energy $LI^2/2$ of an inductor with a current I to the energy calculated from physical quantities of the structure [78]. The energy of a current flowing through a superconducting structure has three components. The first is the energy stored in the magnetic field outside of the superconductor. Considering a perfect electrical conductor, the magnetic field is stored only outside the conductor. Hence, its energy depends merely on the geometry of the structure. The associated inductance is called the external magnetic inductance L_{ext} . We calculated L_{ext} of the structure presented in Section 4.1, modeled as a perfect electrical conductor, using the 3D electromagnetic simulation tool Ansys Q3D Extractor®. The second component is the energy stored in the magnetic field penetrating the superconductor on a length scale λ , yielding the internal magnetic inductance

$$L_{int} = \frac{\mu_0 \lambda}{8} \frac{\sinh(d/\lambda) - d/\lambda}{\sinh^2(d/2\lambda)} \quad (\text{A.53})$$

in the case of a superconducting strip with a current-density distribution according to Eq. A.52. The third component of the energy is the kinetic energy of the charge carriers flowing through the superconductor, with the associated kinetic inductance

$$L_{kin} = \frac{\mu_0 \lambda}{8} \frac{\sinh(d/\lambda) + d/\lambda}{\sinh^2(d/2\lambda)}. \quad (\text{A.54})$$

A derivation of L_{kin} and L_{int} can be found in Ref. [216]. Adding L_{kin} and L_{int} yields the surface inductance

$$L_s = \frac{\mu_0 \lambda}{2} \coth(d/2\lambda), \quad (\text{A.55})$$

which accounts for the inductive behavior of the surface of the superconducting strip. Figure A.7 shows a comparison of the contributions to L_s in dependence of the film thickness. The film with $d = 300$ nm approaches the limit of a thick film $d \gg \lambda$ (see Eq. A.57), although $d/\lambda \approx 4$ only.

Surface resistance

Although the DC resistance vanishes in a superconductor, the AC resistance does not. This is due to the fact that an AC field in a superconductor excites both Cooper pairs and quasiparticles, and the latter cause dissipation. An equivalent resistance

R_s , called the surface resistance, accounts for this dissipation. For a superconducting strip as defined above, the surface resistance is [216]

$$R_s = \frac{\mu_0^2 \sigma_1 \omega^2 \lambda^3}{8} \frac{\sinh(d/\lambda) + d/\lambda}{\sinh^2(d/2\lambda)} \quad (\text{A.56})$$

and is plotted in Fig. A.7.

Surface impedance

Combining R_s and L_s leads to the surface impedance $Z_s = R_s + j\omega L_s$, describing the behaviour of AC fields penetrating into the surface of a superconductor. In the thick-film limit $d \gg \lambda$,

$$Z_s = R_s + j\omega L_s = \frac{1}{2} \sqrt{\frac{j\omega\mu_0}{\sigma_1 - j\sigma_2}} \approx \sqrt{\frac{\omega\mu_0}{\sigma_2}} \left(\frac{\sigma_1}{2\sigma_2} + j \right), \quad (\text{A.57})$$

where $\sigma_1 \ll \sigma_2$ was used. This result is one half of the surface impedance of a semi-infinite halfspace [78], which possesses only one surface. Note that Z_s of a microstrip (strip above a ground plane) is also twice higher than that of an isolated strip as treated here, since most of the current in a microstrip flows on the inner surface next to the ground plane, so the effective cross-sectional area is twice lower. However, in the thin-film limit $d \ll \lambda$ the current density across the thickness of the strip is approximately uniform, and

$$Z_s = \frac{1}{(\sigma_1 - j\sigma_2)d} \approx \frac{\sigma_1}{d\sigma_2^2} + \frac{j}{d\sigma_2}. \quad (\text{A.58})$$

is identical for both isolated strip and microstrip. Figure A.7 illustrates both limits of Z_s .

Surface impedance of the center conductor

In the design of Samples A–C (Sec. 4.1), the effective aspect ratio $\tilde{\ell} = \ell/w$, referred to as a number of squares $\square$, is dominated by the geometry of the meander inductor. The strip of the meander inductor was planned to have a width $w = 1.2 \mu\text{m}$, which reduced due to overetching to $w \approx 1.1 \mu\text{m}$, measured via scanning electron microscopy. Noting that $w \approx 3.7d$, the condition $w \gg d$ is not strictly fulfilled, so J is expected to vary significantly across the width, and Z_s places an upper bound. To estimate a lower bound we take the effective width $\tilde{w} = w + d$. Taking these two limits, one unit cell has $\tilde{\ell} = 129\text{--}164 \square$, giving a surface inductance $L_{s,\text{cell}} = \tilde{\ell} L_s = 5.5\text{--}7 \text{ pH}$, roughly 10% of the total inductance of a meander inductor.

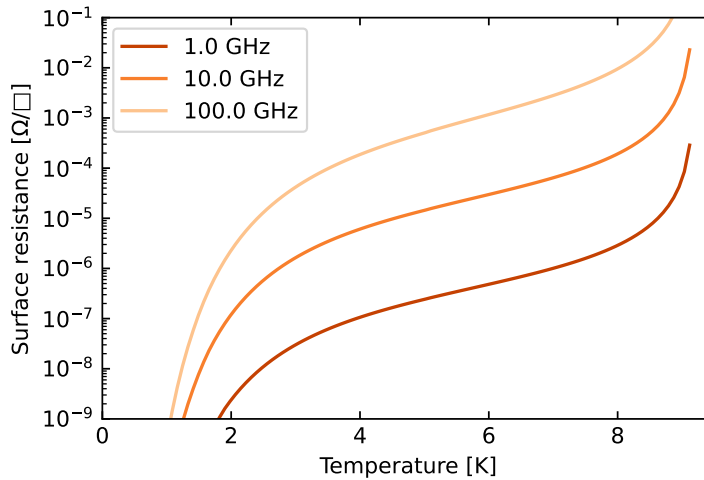


Fig. A.8.: Temperature dependence of surface resistance for different frequencies.

The surface inductance is nearly frequency-independent for $f \ll 2\Delta/h \approx 700$ GHz, and increases less than 4% from 0 K to 4.2 K.

On the other hand, the surface resistance has a strong frequency and temperature dependence, depicted in Fig. A.8, stemming from its quasiparticle nature. At 4.2 K, where a significant number of Cooper pairs are split into quasiparticles, the surface resistance results in ca. 15% of the total loss in a TWPA at 10 GHz, considering the circuit parameters of Sec. 4.2.5. The contribution of the subgap resistance, also being a manifestation of quasiparticle loss, is of the same order. At lower temperatures, however, quasiparticles freeze out quickly, making their contribution to losses negligible even at 2 K.

Contribution of the ground planes to the surface impedance

To analyze the contribution of the ground planes, which constitute the current return path, to the surface impedance, we first examine the current distribution in the ground planes and the center conductor. Charge conservation links the current distribution to the charge distribution caused by electrostatic (capacitive) effects between the center conductor and the ground planes. In the case of a CPW, where inductive and capacitive effects go hand in hand, the current distributions in both the center conductor and the ground planes are nonuniform across their width and increase strongly towards the edges of the conductors. The nonuniform current density must be taken into account when calculating the surface impedance, which can be done by conformal-mapping techniques [217]. In a CPW with a center-conductor width of $w = 32$ μm and a spacing of the ground planes of $s = 64$ μm , the ground planes contribute roughly 30% to the surface impedance.

However, in the lumped-element ladder line the situation is different. The stray capacitance of the center conductor to the ground planes is much smaller than that of the lumped plate capacitors. Neglecting the effects of the stray capacitance to the charge distribution, the current distribution of the center conductor can be seen as uniform across its width (as it was assumed above), and the current distribution in the ground planes is not as strongly confined to their edges as it is the case in a CPW. Hence, the effective cross-sectional area of the ground planes is much larger than that of the center conductor, and $Z_{s,ground} \ll Z_{s,center} \approx Z_s$, where the respective indices denote ground planes and center conductor. Using the formulas of Ref. [217, page 910] for a CPW with $w = 1.1 \mu\text{m}$ and $s = 64 \mu\text{m}$, ignoring that their condition of $d \ll w$ is not strictly valid, the ground plane contributes only 2% to the surface impedance and can be therefore neglected.

A.3 Coplanar-waveguide design and characterization

CPW design

In the TWPA samples, CPW segments are used to route between bonding pads and the TWPA array, as well as for 180° curves connecting two adjacent arrays. These CPWs are designed to have a nominal characteristic impedance of 50Ω . A sketch is shown in Fig. A.9. The center conductor has a width of $w = 32 \mu\text{m}$ and is separated by gaps of $s = 16 \mu\text{m}$ from the ground planes.

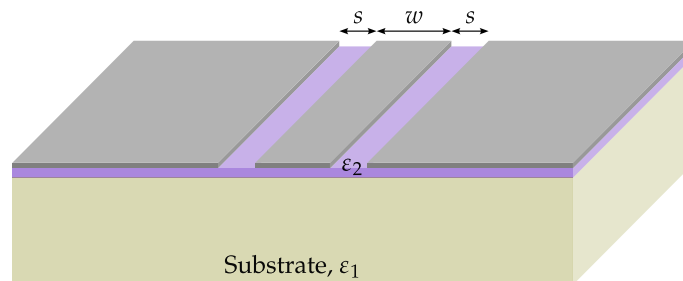


Fig. A.9.: Superconducting coplanar waveguide (CPW), not to scale. The dimensions of the CPW are $w = 32 \mu\text{m}$ and $s = 16 \mu\text{m}$. A thin layer of SiO_x with dielectric constant ϵ_2 is situated between the CPW and the Si substrate.

The conductors are realized in the Nb wiring layer (see Fig. 4.1f for the layer stack) with a thickness $d = 300 \text{ nm}$. The layer stack is grown on a silicon substrate with a thickness of $380 \mu\text{m}$ and a dielectric constant of ca. $\epsilon_1 = 11.9$. Below the conductors are two layers of SiO_x , i.e., the PECVD and thermal SiO_x layers, with a total thickness of about 440 nm and an expected dielectric constant of $\epsilon_2 = 3.9$, deviating significantly from ϵ_1 . Therefore, the interposed SiO_x layer places a

deviation from the simpler case having a uniform substrate with a single dielectric constant ε .

In the limit of infinite ground planes, thin conductors $d \ll w, s$, and a uniform substrate with a single dielectric constant ε , the capacitance and inductance per unit length of a CPW can be calculated analytically as [85]

$$C' = 4\varepsilon_0\varepsilon_{\text{CPW}} \frac{K(k)}{K(k')}, \quad (\text{A.59})$$

$$L' = \frac{\mu_0}{4} \frac{K(k')}{K(k)}, \quad (\text{A.60})$$

with the vacuum permittivity $\varepsilon_0 = 8.854 \times 10^{-12}$ As/Vm, the complete elliptic integral of the first kind $K(k)$, the modulus $k = w/(w + 2s)$, and $k' = \sqrt{1 - k^2}$. The effective dielectric constant is $\varepsilon_{\text{CPW}} \approx (\varepsilon + 1)/2$ in the case of vacuum ($\varepsilon_{\text{vac}} = 1$) in the upper half-plane and ε in the lower half-plane. The influence of the surface impedance (see above) is neglected. The resulting characteristic impedance is

$$Z_{\text{CPW}} = \sqrt{\frac{L'}{C'}}. \quad (\text{A.61})$$

The inductance per unit length, depending only on conductor geometry, is $L' = 0.402$ pH/ μm for the present CPW design. In the case of the thin interposed dielectric layer with $\varepsilon_2 < \varepsilon_1$, lower values than $\varepsilon_{\text{CPW}} \approx (\varepsilon_1 + 1)/2$ are expected. To determine ε_{CPW} , and hence the capacitance per unit length, we follow an experimental approach.

CPW resonator

To determine the characteristics of the CPW such as the capacitance per unit length, and consequently, the characteristic impedance and the phase velocity, we designed a CPW resonator. It is implemented as a capacitively coupled half-wavelength resonator with mean length $\ell = 16.083$ mm. Coupling to the input and output feedlines with impedance Z_0 is realized by symmetric coupling capacitors formed by 10 μm wide gaps in the center conductor of the CPW (Fig. A.10). The resonator sample is co-fabricated with the TWPA samples on the same wafer.

The first three resonant modes were measured at 4.2 K using the experimental setup shown in Fig. 4.4 applying TRL calibration. The $|S_{21}|$ data in the vicinity of the resonances is fitted to

$$|S_{21}| = \left| \frac{T_0}{1 + 2jQ_L \frac{f - f_n}{f_n}} + A_F e^{j\phi_F} \right|, \quad (\text{A.62})$$

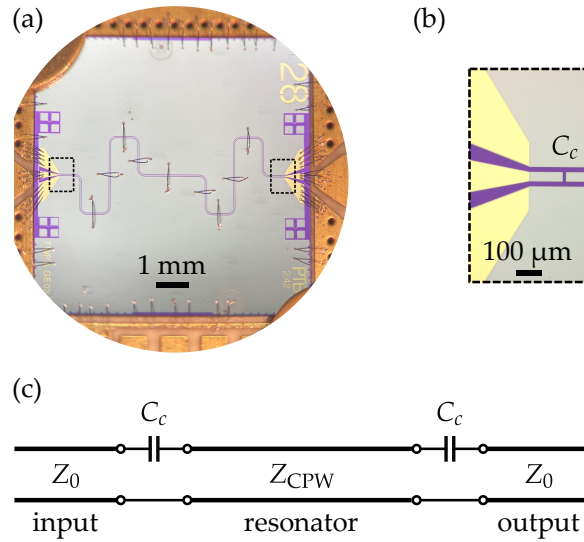


Fig. A.10.: CPW resonator. **(a)** Resonator sample wirebonded to a PCB. Additional wirebonds across the CPW resonator prevent slotline modes by unequal ground-plane potentials. The resonator is capacitively coupled to the feedlines at both ends (dashed boxes). **(b)** Close-up view on one of the coupling capacitances, implemented as a 10 μm gap in the CPW center conductor. **(c)** Circuit diagram of the capacitively coupled transmission-line resonator.

which is a Lorentzian lineshape interfering with a parasitic background path (e.g. cavity modes in the sample package), resulting in an asymmetric Fano resonance [218]. Here, T_0 is the insertion loss at resonance, f_n is the resonance frequency at mode index n , and Q_L is the loaded quality factor. The second term in the equation represents the background path. A least-squares fit yields parameters $f_n, Q_L, T_0, A_F, \phi_F$ for each resonant mode. The loaded quality factor

$$Q_L^{-1} = Q_c^{-1} + Q_i^{-1} \quad (\text{A.63})$$

is given by the coupling quality factor Q_c and the internal quality factor Q_i . The latter represents losses in the CPW and is not further investigated here. The former can be inferred from the insertion loss,

$$Q_c = \frac{Q_L}{T_0}, \quad (\text{A.64})$$

and represents the loss channel due to coupling to the input and output lines. It yields the coupling capacitance [115]

$$C_c = \sqrt{\frac{n\pi}{Q_c}} \frac{1}{4\pi Z_0 f_n}. \quad (\text{A.65})$$

If C_c is neglected, modes are resonant at multiples of half wavelengths, $n\lambda/2 = \ell$. Thus, the effective dielectric constant can be found directly from the resonance frequency as

$$\varepsilon_{\text{CPW}} = \left(\frac{cn}{2\ell f_n} \right)^2, \quad (\text{A.66})$$

where c is the speed of light. Taking C_c into account, the coupling capacitors increase the total capacitance in a lumped-element picture of the resonator, slightly decreasing the resonance frequencies,

$$\frac{f_n(C_{c,\text{eff}})}{n} = \frac{1}{2\pi\sqrt{L_r(C_{r,n} + 2C_c)}} \quad (\text{A.67})$$

in the case of small and symmetric coupling capacitances $C_c \ll (2\pi f_n Z_0)^{-2}$. The effective resonator capacitance $C_{r,n}$ is eliminated using $f_n \approx 1/2\pi\sqrt{L_r C_n}$ and Eq. A.66, and L_r can be found from Eq. A.60. Finally, the corrected value for ε_{CPW} is

$$\varepsilon_{\text{CPW}} = \left(\frac{cn}{2\ell f_n} \right)^2 \left(1 - \frac{4\ell f_n^2 \mu_0 K(k') C_c}{n^2 K(k)} \right) \quad (\text{A.68})$$

and, using Eqs. A.59–A.61, the impedance of the CPW is

$$Z_{\text{CPW}} = \sqrt{\frac{L'}{C'}} = \frac{K(k')}{K(k)} \sqrt{\frac{\mu_0}{16\varepsilon_0 \varepsilon_{\text{CPW}}}}. \quad (\text{A.69})$$

Table A.2 summarizes the results for the first three resonant modes, showing consistent values on C_c , Z_{CPW} and ε_{CPW} . The fourth and fifth resonance are also visible, but overlap with cavity resonances, so their characteristics cannot be determined accurately.

Taking the mean values, the results of this experiment are the CPW characteristics $Z_{\text{CPW}} = 50.22 \, \Omega$ and $\varepsilon_{\text{CPW}} = 5.756$.

Tab. A.2.: Measurement results of the CPW resonator.

Resonance mode n	1	2	3
Resonance frequency	3.8802 GHz	7.7581 GHz	11.636 GHz
Loaded quality factor Q_L	19,000	12,400	7,300
Insertion loss T_0	12.14 dB	9.92 dB	10.0 dB
External quality factor Q_e	77,000	39,000	23,000
Internal quality factor Q_i	25,000	18,000	11,000
Coupling capacitance C_c	2.62 fF	2.61 fF	2.76 fF
Effective permittivity ε_{CPW}	5.754	5.756	5.757
Characteristic impedance Z_{CPW}	50.23 Ω	50.22 Ω	50.21 Ω

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Christoph Kissling

Erklärung

Ich versichere, dass ich die vorliegende Arbeit ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel angefertigt habe. Die aus anderen Quellen direkt oder indirekt übernommenen Daten und Konzepte sind unter Angabe der Quelle gekennzeichnet.

Bei der Auswahl und Auswertung folgenden Materials haben mir die nachstehend aufgeführten Personen in der jeweils beschriebenen Weise unentgeltlich geholfen:

- **Victor Gaydamachenko:** Layout und Herstellung aller in dieser Arbeit diskutierten Proben. Parametrisierung sowie Entwicklung der ersten Version des Simulationsmodells des mPCV-TWPA-Konzepts (Sec. 3.2.4). Das Python-Programm zur automatisierten WRspice-Simulation wurde in enger Zusammenarbeit mit Victor Gaydamachenko entwickelt. Messungen wurden in enger Zusammenarbeit durchgeführt (Sections 4.3, 4.5, 4.6.5, Appendix A.2.1).
- **Lukas Grünhaupt:** Korrekturlesen der vorliegenden Arbeit, Diskussion von Simulations- und Messergebnissen, Automatisierung der Messaufbauten.
- **Alexander B. Zorin:** Konzept und Unterstützung bei der Parametrisierung des mPCV-TWPAs (Sec. 3.2.4).
- **Ralf Dolata:** Entwicklung des ersten WRspice-Simulationsmodells eines RPM-TWPAs, auf welchem das von mir weiterentwickelte RPM-TWPA-Konzept (Sec. 3.2.2) basiert.
- **Marat Khabipov:** Messungen an Josephson-Kontakten in enger Zusammenarbeit (Appendix A.2.1), Design des CPW-Resonators (Appendix A.3).
- **Michel Schröder:** Design und Fertigung der Temperatur-variablen Plattform für die thermische Rauschquelle (Sec. 4.6.2) im Rahmen einer von mir betreuten Techniker-Abschlussarbeit. Design und Fertigung von Probengehäusen und verschiedenen anderen mechanischen Komponenten der Messaufbauten.

- **Antoine Andrieux:** Anpassung und Weiterentwicklung bestehender Simulationsmodelle sowie Durchführung von Simulationen, basierend auf der von mir durchgeführten Entwicklung und Voruntersuchungen des Apodizationskonzeptes (Sec. 3.3.4) im Rahmen eines von mir betreuten Praktikums inklusive Abschlussarbeit. Vorschlag der auf der Fermi-Funktion basierenden Apodizationsfunktion (Eq. 3.38).

Weitere Personen waren an der inhaltlich-materiellen Erstellung der vorliegenden Arbeit nicht beteiligt. Insbesondere habe ich hierfür nicht die entgeltliche Hilfe von Vermittlungs- bzw. Beratungsdiensten (Promotionsberater oder anderer Personen) in Anspruch genommen. Niemand hat von mir unmittelbar oder mittelbar geldwerte Leistungen für Arbeiten erhalten, die im Zusammenhang mit dem Inhalt der vorgelegten Dissertation stehen.

Die Arbeit wurde bisher weder im In- noch im Ausland in gleicher oder ähnlicher Form einer Prüfungsbehörde vorgelegt.

Ich bin darauf hingewiesen worden, dass die Unrichtigkeit der vorstehenden Erklärung als Täuschungsversuch bewertet wird und gemäß § 7 Abs. 10 der Promotionsordnung den Abbruch des Promotionsverfahrens zur Folge hat.

Braunschweig, den 24.06.2024

Christoph Kissling



Die Physikalisch-Technische Bundesanstalt, das nationale Metrologieinstitut, ist eine wissenschaftlich-technische Bundesoberbehörde im Geschäftsbereich des Bundesministeriums für Wirtschaft und Energie.



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