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Measurement uncertainty contribution in the quantisation of measurement values

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Foreword

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The DKD expert reports are to present significant aspects from the field of calibration. The publication of these reports by the DKD will make them accessible to the large community of calibration laboratories both nationally and internationally.

This expert report was approved by the Board of the DKD in December 2019.

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1 Summary

This report deals with the estimation of the measurement uncertainty contribution by way of quantisation for digitally indicating measuring instruments. Its aim is to describe quantisation processes in the best possible way based on the available information, and to make use of this information in practice. In addition to the usual consideration of the uncertainty influence of the readability of the last digit of a display value as a rectangular distribution within the limits of $a_{\mathrm{R}} = \pm 0.5\, Digit$, this report is intended to provide information to better describe the quantisation process from a technical point of view.

The profound analysis of the measurement uncertainty of a digitally indicating measuring device shows that it makes sense to apply a triangular distribution density with a width $2a_{\mathrm{D}}$ of two quantisation steps (code jumps) for the entire measurement process. This yields a combined standard deviation (combined standard uncertainty) $\sigma_{\mathrm{D}} = u_{\mathrm{MU}}(\delta U_{\mathrm{KombD}}) = 2a_{\mathrm{D}} \cdot 0.41$.

Assuming a normal distribution instead of a triangular distribution as an approximation, the estimated combined standard measurement uncertainty would be slightly too large. For this case, the standard deviation is $\sigma_{\mathrm{N}} = u_{\mathrm{MU}}(\delta U_{\mathrm{KombN}}) = 2a_{\mathrm{N}} \cdot 0.5$ (with a coverage probability of 95 %), again with a width of distribution $2a_{\mathrm{N}}$ of two quantisation steps.

This report represents a kind of alternative approach or consideration to the methods used in practice when dealing with quantisation processes (or rounding). It is not so much to be understood as a binding procedure or replacement of previous estimates.

2 Measurement uncertainty contribution in the quantisation of measurement values

2.1 Preliminary considerations

To illustrate the principle of quantisation, let's take an analogue display instrument for voltages – with pointer and lines as scale. And let us assume that the deflection of the pointer is an error-free image of the applied voltage, without frictional influences or non-linearities of the moving-coil system, etc. An applied voltage is visually perceived as an interpolation between two adjacent scale lines. The uncertainty of the reading due to the design of the instrument (mirror scale, pointer, inaccurate interpolation, etc.) shall not be further considered. However, to remain with the example, the following question should be raised: what about the accuracy and precision of the lines of the scale? In other words, a corresponding uncertainty of the reading is to be assumed for the resulting indication value. Hence, an uncertainty contribution of the measurement result due to inaccurate placement of the scale lines must be assumed. This could be called "scale uncertainty".

Completely separate from this, the reading could **roughly** be arranged in such a way that the value to which the lower scale line is assigned is interpreted as the result of the measurement; this would correspond to rounding (down). This naturally implies an uncertainty of the measurement result, given that all applied voltage values between two adjacent scale lines will always yield just this one measurement result. This uncertainty could be called "rounding uncertainty".

These preliminary considerations have been taken into account in the consideration of the measurement uncertainty of the quantisation.

2.2 General information

A digital voltmeter with the transmission factor $V = 1$ (the output voltage or indicated voltage U_{Anz} divided by the applied input voltage U_{Ein}) will be used as an illustrative example.

2.2.1 Assumptions for a "flawless" quantisation

- "No missing code": Any possible display value can be obtained by applying a voltage.
- Monotonicity: With increasing (decreasing) values of the applied voltage, quantisation jumps of the indicated voltage only occur in ascending (descending) order.
- The hysteresis in case of decreasing voltage (i.e. when reversing the applied voltage) is considered to be insignificantly small.
- The change of the indicated voltage that is required for a code jump is everywhere the same.
- The deviations of the indication that can be determined by means of calibration are to be compensated by a correction.

- The way the analogue/digital conversion is realised in a measuring instrument with digital display does not play a significant role in the analysis of the measurement uncertainty for quantisation (dual-slope integration method, parallel method, compensation method, weighing method, etc.).

2.2.2 Terms and definitions

Quantisation
In digital signal processing, quantisation is a mapping that occurs when analogue signals are digitised.
A continuously detectable measurement value is converted into a "digitised" form and can thus be represented as a number with a certain number of digits (with additional indication of the unit of the measurement value).

Digitisation
Digitisation is generally understood as the preparation of information (the process of changing data into a digital form) for processing or storage in a digital technical system. The original information is available in any type of analogue form and is then converted into a digital signal – possibly over several stages – merely consisting of discrete values.

Classification (Classing)
Classification refers to the division of statistical series into separate classes of size. Depending on its value on the corresponding variable, each element is assigned to exactly one class. All values of a class lie within the upper and the lower limits of a class; the difference between the upper and the lower limit of the class is the class width. In this report, the classification of the measurement values is equivalent to a rounding.

Discretisation
Representation of a function $y(x)$ by values $y_i(x_i)$ at a finite number of sampling points x_i.
(The term "discretisation" is not relevant in this report. It is related, for example, to the term "sampling" and must be distinguished from the terms "digitisation" and "quantisation").

Quantisation step (code jump; size of the jump)
Step height of a (the smallest possible) digit (digital step) in the indication. The corresponding value will be referred to in this report as ΔU_{Code}.

Switching threshold (jump discontinuity; switching point)
Value of the measured variable causing the digital display to switch from one digit to another.

Expanded measurement uncertainty U_{MU}
Since in this report voltage values are referred to as U, the expanded measurement uncertainty which is usually also represented by $U(\ldots)$ is to be given the index "MU", i.e. U_{MU}. By using an additional index, it is possible to indicate if a specific expanded measurement uncertainty is meant.

Width of a distribution density
Here indicated as $2a$, i.e. a as half-width of the distribution or the distribution density function. Outside of this half-width there are normally no values of the function.

Scale [VIM 3.5]
Part of a displaying measuring instrument, consisting of an ordered set of marks together with any associated quantity value. In this report the markings are called scale points.

Scale interval
The respective range defined by two adjacent scale points.

Note regarding the use of the term rounding in this report:
In the broadest sense, quantisation in digital display devices is a rounding of the measurement values on the display (indicated values). In a measuring device, it is realised in a specific way as a design feature (rounding up, rounding down or rounding in the strict mathematical sense, or commercial rounding). Frequently, details are not known and are usually not given in the technical data. However, the type of rounding has a slight influence on the measurement result. After calibration, this influence is eliminated by correction.

Rounding down, which is frequently encountered in quantisation, is assumed in the present report. Technically, it is probably most easily realised, and it is also easily explained (cutting off the surplus digits).

It is important to mention that the type of rounding (rounding down, rounding up or commercial rounding) does not contribute to the measurement uncertainty – always provided that the assumed correction of the measurement result has been carried out properly.

2.3 Model of evaluation for the indication of a digital voltmeter

The model equation (model of evaluation) for the indication of a digital voltmeter is given as follows:

$$U_{Anz} + \delta U_{Komb} = U_{Ein} + \Delta U_{Anz} + \delta U_{Skale} + \delta U_{Klass} + \delta U_{Sonst}$$

(Equation 1)

U_{Anz}	Indication value of the digital voltmeter
U_{Ein}	Voltage applied to the input of the digital voltmeter
ΔU_{Anz}	Indication error (display deviation), e.g. determined by calibration
δU_{Komb}	Unknown or indeterminably small deviation carrying an uncertainty attributed to the indication value of the digital voltmeter
$\delta U_{\mathrm{Skale}}$	Unknown or indeterminably small deviation of the position of the scale points being the cause of a corresponding uncertainty (see below)
δU_{Rund}	Unknown or indeterminably small deviations due to rounding down, i.e. allocation to scale intervals, which are the cause of a corresponding uncertainty (see below)
$\delta U_{\mathrm{Sonst}}$	Placeholder for other unknown or indeterminably small deviations that may occur during measurement with a digital voltmeter and may be the cause of further uncertainty contributions. They will not be considered in this report.

2.3.1 Measurement uncertainty contribution due to the unknown position of the scale points

The digital voltmeter includes a scale against which the input voltage of the voltmeter is compared. Thus, an internal mapping of the input voltage is available as a reference to the almost correct scale points which were calibrated and adjusted as well as possible during production. However, when comparing the position of the realised scale points with a reference scale that represents "correct" values of the position of the scale points, certain deviations $\delta U_{\text{Skale}}(i) = U_{\text{Skale}}(i) - U_{\text{Ref}}(i)$ will be detectable in each scale interval i (see Figure 1, upper part).
Using a sufficiently accurate adjustable calibrator, it would in theory even be possible to determine these deviations for each scale interval. However, this would be very time-consuming and is hardly ever done. On the other hand, this would provide a statistical indication of the scale deviation of the specific voltmeter in question.
A common and useful practice is to make a general statistical statement regarding the deviations $\delta U_{\text{Skale}}(i)$ in order to obtain an estimate of the corresponding measurement uncertainty contribution. Generally, the following information is available:

1. The deviation $\delta U_{\text{Skale}}(i)$ can be max. plus or minus half the width of a scale interval, i.e. a total of one scale interval width. With the scale interval width $\Delta U_{\text{Skale}}(i)$, this can be specified as $-0.5 \cdot \Delta U_{\text{Skale}}(i) \leq \delta U_{\text{Skale}}(i) < +0.5 \cdot \Delta U_{\text{Skale}}(i)$. This applies to all scale intervals i. The half-width of the distribution density function is $a = \Delta U_{\text{Skale}}(i)/2$.
2. Since information on the distribution density of the deviation $\delta U_{\text{Skale}}(i)$ is usually not available, only a uniform distribution can be assumed.

The contribution to the measurement uncertainty due to the possible deviation of the position of the scale points is therefore:

$$u_{MU}(\delta U_{Skale}) = \frac{0.5 \cdot \Delta U_{Skale}(i)}{\sqrt{3}} \approx \frac{0.5 \cdot \Delta U_{Code}}{\sqrt{3}}$$

(Equation 2)

on the assumption of a uniform distribution.

As a statistical quantity, the distance of a scale point $\Delta U_{\text{Skale}}(i)$ is generally not precisely known, but on average $\Delta U_{\text{Skale}}(i) \approx \Delta U_{\text{Code}}$, with the quantisation level ΔU_{Code} being determined by the hardware of the digital voltmeter; its value can be used in equation 3.

2.3.2 Measurement uncertainty due to rounding of the measured value

As described, an input value falls within a defined scale interval of the internally realised scale of the voltmeter and is then rounded down[1] according to the "scale points" of the internal scale. The rounded value $U_{\text{Rund}}(i)$ thus leads to the digital indication. Intermediate values which, for example, might be obtained through interpolation, are suppressed by the rounding process. This classification (classing) of the measured values is the actual quantisation process. Obviously, the original input voltage is displayed somewhat coarser (less accurate). This leads to an unknown or indeterminably small deviation which is presented as $\delta U_{\text{Rund}}(i) = U_{\text{Rund}}(i) - U_{\text{Ein}}$ (see Figure 1, lower part). In other words, all input voltage values that are within a certain rounding interval will produce the same indication (same display value). Here, it is no longer possible to determine the magnitude of the

[1] technical design feature

difference $\delta U_{\mathrm{Rund}}(i)$. A measurement uncertainty contribution must therefore be applied. The width of the distribution density function is $2a_{\mathrm{R}} = \Delta U_{\mathrm{Rund}}(i) \approx \Delta U_{\mathrm{Code}}$.
Since an input voltage can have any value within the measuring range, the same is true for a rounding interval, where any value can occur, with a uniform distribution.
Thus, the contribution to the measurement uncertainty due to rounding is:

$$u_{MU}(\delta U_{Rund}) = \frac{0.5 \cdot \Delta U_{Code}}{\sqrt{3}}$$

(Equation 3)

on the assumption of a uniform distribution.

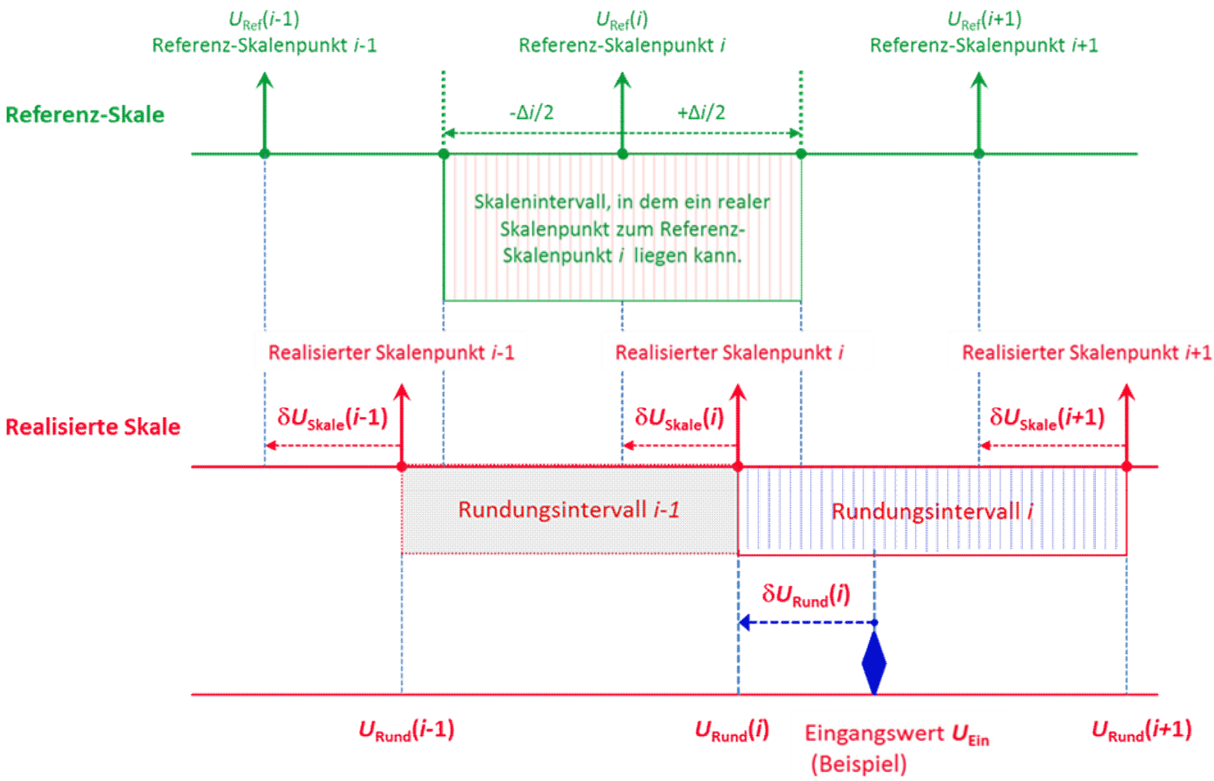

Figure 1: Scale section

Referenz-Skalenpunkt = reference scale point
Referenz-Skale = reference scale
Skalenintervall, in dem ein realer Skalenpunkt zum Referenz-Skalenpunkt *i* liegen kann = Scale interval in which an actual scale point may be located with respect to the reference scale point *i*
Realisierter Skalenpunkt = realised scale point
Realisierte Skale = realised scale
Rundungsintervall = rounding interval
Eingangswert = input value / Beispiel = example

2.3.3 Measurement uncertainty budget

It is to be noted that the two contributions

1. standard uncertainty contribution (scale uncertainty) due to the possible deviation of the position of the scale points $u_{\mathrm{MU}}(\delta U_{\mathrm{Skale}})$ and
2. measurement uncertainty (inaccuracy of the rounding) due to rounding $u_{\mathrm{MU}}(\delta U_{\mathrm{Rund}})$

are independent from each other, i.e. uncorrelated.

The combined standard uncertainty from both contributions is:

$$u_{\mathrm{MU}}(\delta U_{\mathrm{Komb}}) = \sqrt{(u_{\mathrm{MU}}(\delta U_{\mathrm{Skale}}))^2 + (u_{\mathrm{MU}}(\delta U_{\mathrm{Rund}}))^2}$$

(Equation 4)

Hence:

$$u_{MU}(\delta U_{Komb}) = \frac{\Delta U_{Code}}{\sqrt{6}} = 0.4082 \cdot \Delta U_{Code}$$

(Equation 5)

Unfortunately, this result does not indicate its associated distribution density or the width of its distribution density ($2a_{\mathrm{D}}$). However, this information is of importance for further use of the result. Especially if the two stated uncertainty contributions are of a significant nature and a reliable coverage factor for the determination of the expanded measurement uncertainty of the digital indication is required. This missing information is examined in the following section.

2.3.4 Distribution density function of the combined standard uncertainty

As illustrated, there are two significant contributions to the measurement uncertainty when using a digital voltmeter (see sections 2.3.1 and 2.3.2), namely:

1. measurement uncertainty contribution (scale uncertainty, (equation 2)) due to the possible deviation of the position of the scale points and
2. measurement uncertainty (rounding inaccuracy, (equation 3)) due to rounding: $u_{\mathrm{MU}}(\delta U_{\mathrm{Rund}})$

If both are uniformly distributed, each with a width of one interval $\Delta U_{\mathrm{Code}} \approx \Delta U_{\mathrm{Skale}}(i)$ to determine the distribution density of the combined uncertainty, the mathematical **process of convolution** (see Appendix) must be applied [PAPOULIS]. For the present (simple) case, however, the result can often be found in the corresponding literature [e.g. GUM]. The result of the convolution is a triangular distribution density function with the width $2a_{\mathrm{D}} = 2 \cdot \Delta U_{\mathrm{Code}}$ (double interval width equalling two quantisation steps).

This triangular distribution density function [see GUM] involves a standard deviation σ_{D} (standard measurement uncertainty $u_{\mathrm{MU}}(\delta U_{\mathrm{KombD}})$, also see section 0)

$$u_{\mathrm{MU}}(\delta U_{\mathrm{KombD}}) = \sigma_D = \frac{2 \cdot \Delta U_{\mathrm{Code}}}{2 \cdot \sqrt{6}} = 0.4082 \cdot \Delta U_{\mathrm{Code}}$$

(Equation 6)

(half the width of the distribution density function divided by the root of six in case of triangular distribution)

as well as a coverage factor $k_{\mathrm{D}} = 1.902$.

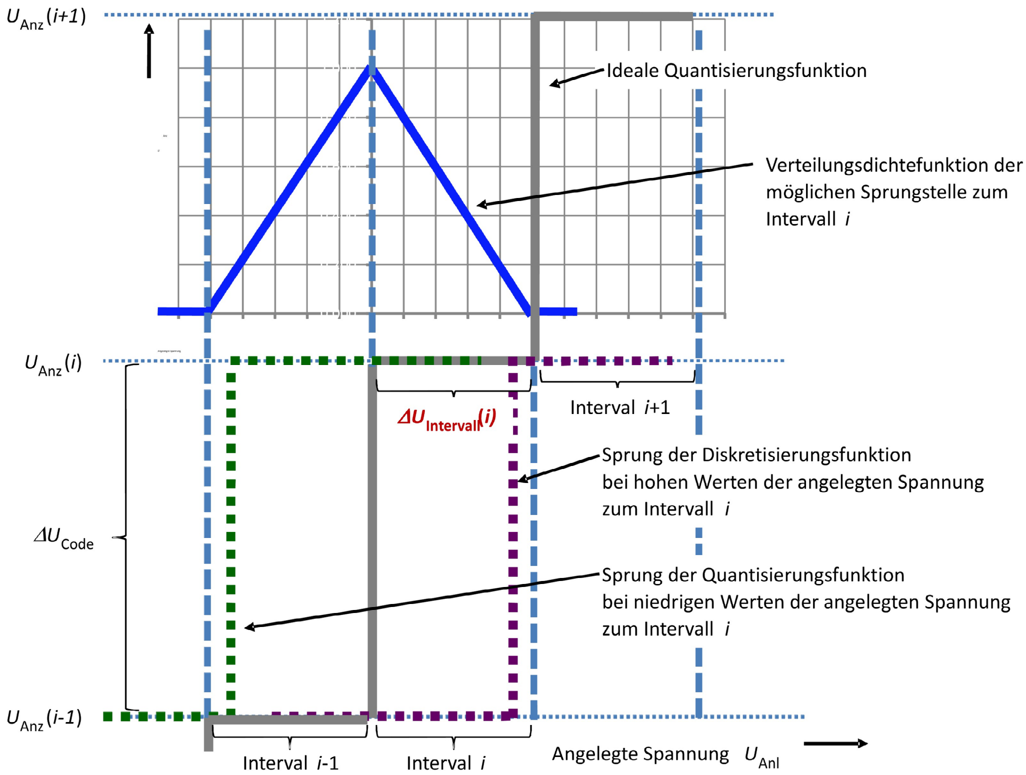

Figure 2: Visualisation of the actual quantisation

Ideale Quantisierungsfunktion = ideal quantisation function
Verteilungsdichtefunktion der möglichen Sprungstelle zum Intervall *i* = distribution density function of the possible jump discontinuity to interval *i*
Sprung der Diskretisierungsfunktion bei hohen Werten der angelegten Spannung zum Intervall *i* = jump of the discretisation function to interval *i* in case of high values of the applied voltage
Sprung der Diskretisierungsfunktion bei niedrigen Werten der angelegten Spannung zum Intervall *i* = jump of the discretisation function to interval *i* in case of low values of the applied voltage
Angelegte Spannung = applied voltage

With some upward room left for the measurement uncertainty, an approximation of the triangular distribution by a normal distribution (for example with a coverage probability of 95.45 % corresponding to $2\sigma_{\mathrm{N}}$) can be justified.

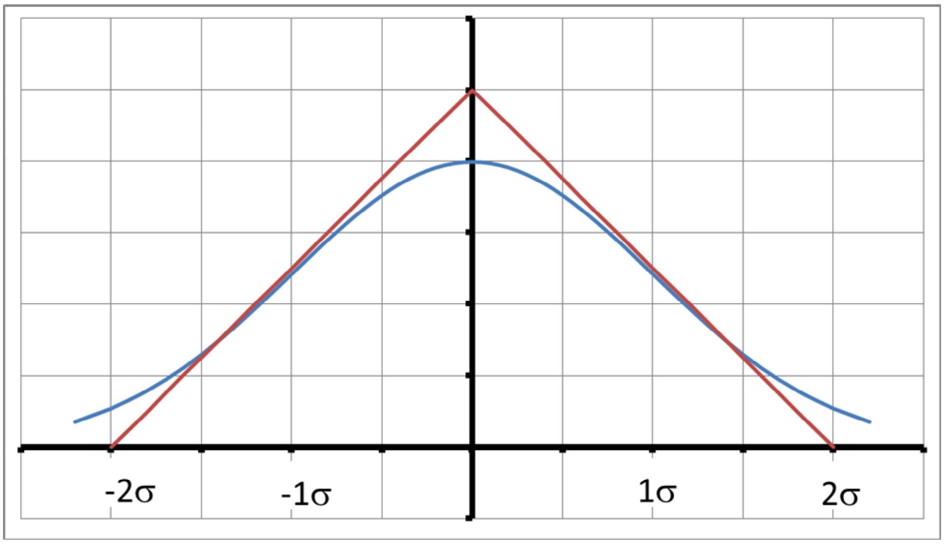

Figure 3: Comparison of the triangular distribution density with that of the normal distribution

This means that the standard uncertainty is calculated with the standard deviation as follows

$$u_{\mathrm{MU}}(\delta U_{\mathrm{KombN}}) = \sigma_N = \frac{2 \cdot \Delta U_{\mathrm{Code}}}{2 \cdot 2} = \frac{\Delta U_{\mathrm{Code}}}{2} = 0.5 \cdot \Delta U_{\mathrm{Code}}$$

(Equation 7)

(half the width of the distribution density function divided by 2 in case of normal distribution)

Here, the coverage factor is $k_{\mathrm{N}} = 2.0$; therefore, the expanded measurement uncertainty is $U_{\mathrm{MU}}(\delta U_{\mathrm{KombN}}) = \Delta U_{\mathrm{Code}}$. In practice, this simple approach is suitable for most applications (see section 2.5).

2.4 Ideal quantisation

However, there are cases of quantisation where no scale is involved; consequently, there is no contribution to the measurement uncertainty due to a possible deviation of the position of the scale points. This means that in the ideal case the quantisation point is

$$U_{\mathrm{Ein}} = U_{\mathrm{Anz}}(i) \;\; @\, i = 1, \dots, n$$

with
n being the number of quantisation steps.

Thus, a certain value of the indication $U_{\mathrm{Anz}}(i)$ may be caused by an applied voltage in the interval $U_{\mathrm{Anz}}(i) \leq U_{\mathrm{Ein}} < U_{\mathrm{Anz}}(i) + \Delta U_{\mathrm{Code}}$, given a uniform distribution in the interval. Due to rounding, there is just this one contribution to the measurement uncertainty. Formally, this corresponds to the mathematical process of rounding down (see section 2.3.2).
The width of the rectangular distribution density is $2a_{\mathrm{I}} = \Delta U_{\mathrm{Code}}$, i.e. **one** quantisation step.
This yields a standard deviation of

$$\sigma_{\mathrm{I}} = 0.5 \cdot \Delta U_{\mathrm{Code}} \cdot 1/\sqrt{3} = 0.2887 \cdot \Delta U_{\mathrm{Code}}$$

(Equation 8)

(half the width of the density function divided by the root of 3 in case of uniform distribution)

The coverage factor is $k_\mathrm{I} = 1.653$.

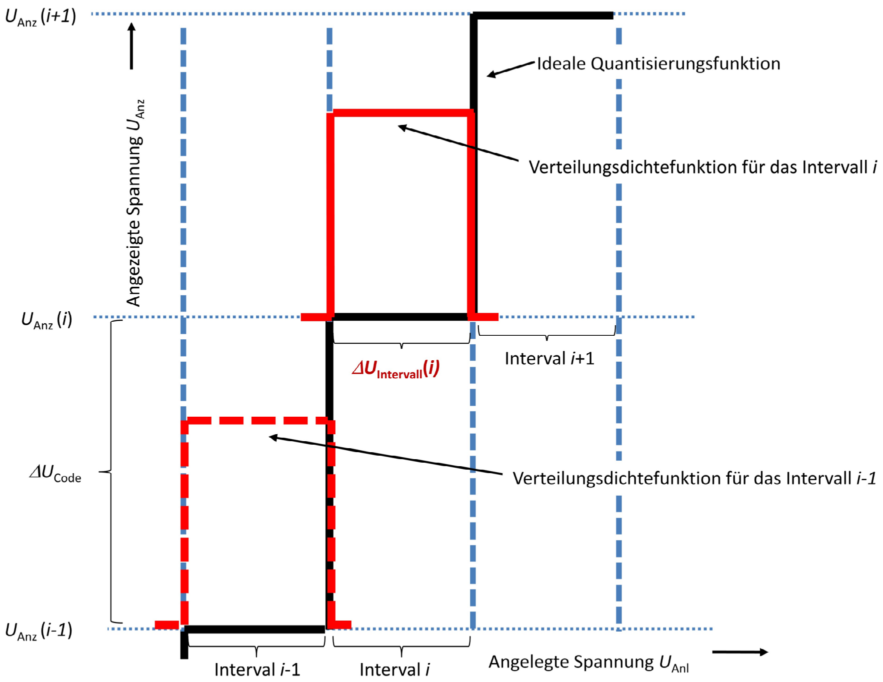

Figure 4: Visualisation of the ideal quantisation process

Ideale Quantisierungsfunktion = ideal quantisation function
Verteilungsdichtefunktion für das Intervall *i* = distribution density function for interval *i*
Angezeigte Spannung = indicated voltage
Angelegte Spannung = applied voltage

2.5 Practical recommendations

There are two suggestions on how to proceed in practice when wishing to (verifiably) estimate the measurement uncertainty contribution for a quantisation process:

1. If requiring a reliable but not an excessively high value for the measurement uncertainty, the triangular distribution is applied over two intervals. Thus the associated standard measurement uncertainty can be calculated as standard deviation with

$$\sigma_\mathrm{D} = \frac{\Delta U_\mathrm{Code}}{\sqrt{6}} = 0.4082 \cdot \Delta U_\mathrm{Code}$$

(Equation 9)

and the coverage factor $k_\mathrm{D} = 1.9$ can be used to determine the expanded measurement uncertainty $U_\mathrm{MU}(\delta U_\mathrm{KombD})$. This would be:

$$U_\mathrm{MU}(\delta U_\mathrm{KombD}) = \sigma_\mathrm{D} \cdot k_\mathrm{D} = 0.7757 \cdot \Delta U_\mathrm{Code} \qquad \text{(Equation 9.1)}$$

This procedure is based on the assumption that the undefined position of the switching point is uniformly distributed within the quantisation intervals. Normally, further information is not known or available. Real A/D converters in digital display units will probably have a more favourable distribution density (if only for manufacturing reasons). **This means that the assumption of a uniform distribution is very conservative, and the actual measurement uncertainty is certainly not underestimated.**

2. If there is some upward room for the measurement uncertainty, an approximation of the triangular distribution by a normal distribution (for example with a coverage probability of 95.45 % corresponding to $2\sigma_{\mathrm{N}}$) may be justified (see Figure 3). This means the standard uncertainty together with the standard deviation results in

$$\sigma_{\mathrm{N}} = \frac{2 \cdot \Delta U_{\mathrm{Code}}}{2 \cdot 2} = \Delta U_{\mathrm{Code}}/2$$

(Equation 10)

The coverage factor is $k_{\mathrm{N}} = 2.0$; thus the expanded measurement uncertainty is $U_{\mathrm{MU}}(\delta U_{\mathrm{KombN}}) = \Delta U_{\mathrm{Code}}$.

In everyday practice, this simple approach is suitable for most applications. It does not overestimate the measurement uncertainty and, with the coverage factor $k_{\mathrm{N}} = 2$ for a normal distribution of a combined measurement uncertainty, it fits well into a superordinate uncertainty analysis for a system with further uncertainty contributions.

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Section 4.3 Type B evaluation of standard uncertainty
Subsection 4.3.9 (Trapezoidal distribution, convolution)

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4 Appendix

4.1 Characteristic data for different distribution density functions

Type of distribution	σ	P_{95}	k-Faktor	U_{MU}
Normal distribution (95.45%)	$\sigma_N = 1/2 = 0.5$	$P_{N95} = 0.9545$	$k_{N95} = 2$	$U_{N95} = 1.000$
Uniform distribution	$\sigma_R = 1/\sqrt{3} = 0.5774$	$P_{R95} = 0.950$	$k_{R95} = 1.653$	$U_R = 0.954$
Triangular distribution	$\sigma_D = 1/\sqrt{6} = 0.4082$	$P_{D95} = 0.776$	$K_{D95} = 1.902$	$U_D = 0.766$
U-shaped distribution	$\sigma_U = 1/\sqrt{2} = 0.707$	$P_{U95} = 0.998$	$k_{U95} = 1.411$	$U_U = 1.000$
Parabolic distribution	$\sigma_P = 1/\sqrt{5} = 0.4472$	$P_{P95} = 0.811$	$k_{P95} = 1.814$	$U_P = 0.363$
Cubic distribution	$\sigma_K = \sqrt{2/15} = 0.365$	$P_{K95} = 0.691$	$k_{P95} = 1.892$	$W_K = 0.363$
Normal distribution (99.00%)	0.388		2	0.766
Normal distribution (99.45%)	0.333 (3σ)		2	0.666
R → N	0.577		2	1.154
D → N	0.408		2	0.816

Table 1: Characteristic data for different distribution density functions (with a = 1, half the width of the distribution density function)

4.2 Convolution

4.2.1 Mathematical definition

Assumption: $F_1(t)$ and $F_2(t)$ are integrable functions for $-\infty < t < +\infty$. The integral:

$F_1(t) * F_2(t) = \int_{-\infty}^{+\infty} F_1(t-\tau) \cdot F_2(t) d\tau$ denotes the convolution of $F_1(t)$ and $F_2(t)$.

4.2.2 Convolution in a game of dice (*Alea iacta est*)

The numbers (of spots) on the sides of a dice (1 to 6) are uniformly distributed. However, the sums of the numbers when tossing two dice are not. When playing with two dice, the following 6 · 6 = 36 combinations are possible:

(1;1), (1;2), (1;3), (1;4), (1;5), (1;6),
(2;1), (2;2), (2;3), (2;4), (2;5), (2;6),
(3;1), (3;2), (3;3), (3;4), (3;5), (3;6),
(4;1), (4;2), (4;3), (4;4), (4;5), (4;6),
(5;1), (5;2), (5;3), (5;4), (5;5), (5;6),
(6;1), (6;2), (6;3), (6;4), (6;5), (6;6),

By forming the respective sums and arranging them systematically, we get the following picture:

Frequency [2)]		**1/36**	**2/36**	**3/36**	**4/36**	**5/36**	**6/36**	**5/36**	**4/36**	**3/36**	**2/36**	**1/36**
							(6;1)					
						(5;1)	(5;2)	(6;2)				
					(4;1)	(4;2)	(4;3)	(5;3)	(6;3)			
				(3;1)	(3;2)	(3;3)	(3;4)	(4;4)	(5;4)	(6;4)		
			(2;1)	(2;2)	(2;3)	(2;4)	(2;5)	(3;5)	(4;5)	(5;5)	(6;5)	
		(1;1)	(1;2)	(1;3)	(1;4)	(1;5)	(1;6)	(2;6)	(3;6)	(4;6)	(5;6)	(6;6)
Sum		**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**

There are 11 different sum values (≠ 0), though with varying frequency. From this, we can discern a distribution density[3)] for the sum of the numbers when rolling two dice, although it is only a discrete one given such a small number of values. The shape is approximately triangular. Its maximum is 7, the mean value of all possible sum values, and it is (almost) twice as wide (11) as the known discrete uniform distribution density ** of the number of points of each of the dice involved (1; 2; 3; 4; 5; 6), i.e. 6 different values in each case.
Here it becomes obvious that the sum of two uniformly distributed (discrete) random values is again a random value, but with a triangular (discrete) distribution density ** of (almost) double width.
Mathematically, this is referred to as “convolution” for two random numbers. Generally, the calculation of a convolution is quite complicated. It is described in publications on stochastic processes [e. g. PAPOULIS].

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2 The frequency values are represented by the divisor 36, which is supposed to formally indicate the scaling, so that their sum is 1.

3 It would be mathematically exact to imagine the discrete distribution density function with a number of Dirac impulses, as “needles” of infinite height, but defined areas, because in the case of rolling a dice the density is naturally only defined for whole numbers.

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